LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 15

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Problem 1 (10*+10*+10*, range and kernel)**.** Let $f : \mathrm{V} \to \mathrm{W}$ be a linear map between vector spaces V and W.

(i) Prove that the kernel of f is a subspace of V.
(ii) Prove that the range of f is a subspace of W.
(iii) Compute $f(0)$.

Problem 2 (20, linear maps)**.** Let $\mathrm{P}(\mathbb{R})$ be the set of polynomials with real coefficients and let $f : \mathrm{P}(\mathbb{R}) \to \mathbb{R}^2$ be a linear map such that on $\{X^3 + X + X^2, 1 + X, X^4 - X, X^4 + X^3\}$ the map has the values

$$f(X^3 + X + X^2) = (1, 4),\ f(1 + X) = (-7, 5),\ f(-X + X^4) = (2, -1),\ f(X^3 + X^4) = (6, -2).$$

Compute $f(4X^4 + 2X^3 + X^2 - 4X - 2)$.

Problem 3 ((10+(5+10+10))+15, transformation matrix)**.** Let $\mathrm{P}_n(\mathbb{R})$ be the set of polynomials of degree not bigger than n.

(i) Consider the map $f : \mathrm{P}_2(\mathbb{R}) \to \mathrm{P}_2(\mathbb{R})$

$$f(aX^2 + bX + c) := (c + b)X^2 + (a + 2b - c)X + c.$$

  (a) Prove that f is linear.
  (b) Consider the bases: $\mathrm{B} = \{1, X, X^2\}$ and $\mathrm{C} = \{1+X^2, 1+X+X^2, X-X^2\}$. Compute the transformation matrices:

$$[f]_{\mathrm{B}},\ [f]_{\mathrm{C}} \text{ and } [f]_{\mathrm{C}\leftarrow\mathrm{B}}.$$

(ii) Consider the linear map $g : \mathrm{P}_3(\mathbb{R}) \to \mathbb{R}^2$ given by

$$g(aX^3 + bX^2 + cX + d) = (a + b + d, c - 2d).$$

Compute the transformation matrix $[g]_{\mathrm{C}\leftarrow\mathrm{B}}$ of g for the bases

$$\mathrm{B} = \{1 + X + X^2, X^2 - 1, X^3 + X^2, X + 1\} \text{ and } \mathrm{C} = \{(1, 1), (1, -1)\}.$$

Problem 4 (10+10+10*+20*, kernel and range)**.** (i) Compute the kernel of the map g in Problem 3(ii).
(ii) Determine if the map f in Problem 3(i) is bijective.
(iii) Find a linear map $h : \mathbb{R}^2 \to \mathrm{P}_3(\mathbb{R})$ such that $g \circ h$ is the identity map of $\mathbb{R}^2$. Compute the range of h. (Remark: There are many choices for h. Just find one.)
(iv) Consider the map $k : \mathrm{P}(\mathbb{R}) \to \mathrm{P}(\mathbb{R})$ defined by

$$k(P(X)) := (P(-X^2))',$$

i.e. after substituting $-X^2$ for X it takes the derivative. Show that k is linear and compute the range of k.

Date: Please hand in before the lecture by **Friday, the 3rd of January 2025**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.