LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 14

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Problem 1 (20, orthogonal reflection). (i) Let $(\mathrm{V}, \langle\ \rangle)$ be a finite dimensional inner product space and $\mathrm{T} : \mathrm{V} \to \mathrm{V}$ be an orthogonal reflection about a linear hyperplane H, i.e.

- T is linear,
- $\mathrm{T}(h) = h$ for all $h \in \mathrm{H}$ and
- $\mathrm{T}(u) = -u$ for all $u \in \mathrm{H}^\perp$.

Prove that there exists a non-zero $u \in \mathrm{V}$ such that

$$\mathrm{T}(v) = v - 2\frac{\langle u, v\rangle}{\langle u, u\rangle}u$$

for all $v \in \mathrm{V}$.

(ii) Consider $(\mathrm{V}, \langle\ \rangle) = (\mathbb{R}^n, \langle\ \rangle_{\mathrm{I}_n})$ and T be as in (i). Let $\mathrm{A} \in \mathbb{R}^{n\times n}$ such that $\mathrm{T} = \mathrm{T}_\mathrm{A}$. (Take $\mathrm{A} = [\mathrm{T}]_{\mathrm{B}_{st}}$.) Prove that $\mathrm{A} = \mathrm{A}^T = \mathrm{A}^{-1}$.

(iii) Let $\phi \in [0, \pi[$ be a fixed angle and put $\mathrm{H} := \mathbb{R}\begin{pmatrix}\cos(\phi)\\ \sin(\phi)\end{pmatrix}$. Let T be the orthogonal reflection about H. Compute $\psi \in]-\pi, \pi]$ such that $\mathrm{T} = \mathrm{T}_\mathrm{A}$ for $\mathrm{A} = \begin{pmatrix}-\cos(\psi) & \sin(\psi)\\ \sin(\psi) & \cos(\psi)\end{pmatrix}$.

Problem 2 (5+5+10+10*+10, Singular value decomposition (SVD)). Find a SVD for the following matrices:

(i) $(1\ 2)$

(ii) $(1\ 2\ 3)^T$

(iii) $\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$

(iv) $\begin{pmatrix}1 & -\frac{1}{5} & \frac{7}{5}\\ -1 & -\frac{1}{5} & \frac{7}{5}\\ 0 & \frac{11}{5} & -\frac{2}{5}\end{pmatrix}$

(v) $\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 1\end{pmatrix}$

Problem 3 (5+10+(15**+5)+12, quadratic forms). Determine if the following given two quadratic forms are equivalent and if they are orthogonally equivalent.

(i) $x_1^2 + x_2^2$ and $x_1^2 + 2x_1x_2 + 2x_2^2$ on $\mathbb{R}^2$,

(ii) $x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_3$ and $2x_1^2 + 2x_2^2$ on $\mathbb{R}^3$,

(iii) $x_1^2 - x_2^2 - x_3^2 - x_4^2$ and $x_1^2 + x_2^2 + x_3^2 - x_4^2$ on $\mathbb{R}^4$.

Determine the definiteness of the 6 given quadratic forms. (The bonus points are for the equivalence question in (iii). Hint: consider inner product spaces and restrict to two dimensional subspaces.)

Problem 4 (20 points, spectral decomposition). Find a spectral decomposition of

$$\begin{pmatrix}-1 & 0 & 1\\ 0 & -2 & 1\\ 1 & 1 & -1\end{pmatrix}.$$

Date: Please hand in before the lecture by **Sunday, 29th of December 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.