LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 12

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Problem 1 (20, finite dimensional inner product space)**.** Let A be a real symmetric matrix of size n. Prove that the following assertions are equivalent:

(i) $\langle , \rangle_{\mathrm{A}}$ is an inner product.
(ii) All eigenvalues of A are positive.

Problem 2 (10+10+10*, infinite dimensional inner product space)**.** We consider on C([0, 1]), the set of continuous real-valued functions on [0, 1], the pairing

$$\langle f, g\rangle := \int_0^1 f(x)g(x)dx.$$

(i) Prove that $\langle , \rangle$ is an inner product.
(ii) We restrict $\langle , \rangle$ on the subset V of all real polynomial functions on $[0, 1]$. Given a non-negative integer n, let p_n be the polynomial function given by $p_n(x) = x^n$, $x \in [0, 1]$. Find
  (a) the orthogonal complement W of p_1 in V. Compute a basis of W.
  (b) the orthogonal complement of $\{p_1, p_3, p_5, \ldots\}$ in V.

Problem 3 (10+10+10*, finite dimensional inner products spaces)**.** Consider the following pairings $\langle , \rangle_{\mathrm{A}}$. Determine if the pairing is an inner product and in any case compute the Gram matrix with respect to the given basis.

(i)
$$\begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}, \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

(ii)
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right\}$$

(iii)
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Problem 4 (30+15*, Gram-Schmidt)**.** Gram-Schmidt orthonormalize the column sets of the following matrices to obtain an orthonormal basis for the column space. Further (*) provide a QR decomposition of the matrix.

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 \\ 2 & -2 & 3 \\ 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 3 & -2 \\ 2 & 2 & -2 \\ -2 & 1 & 1 \\ -1 & -2 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Date: Please hand in before the lecture by **18th of December 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.