LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 11

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Problem 1 (20, multiplicities)**.** Find three real square matrices A_1, A_2, A_3 of size 6 with characteristic polynomial $(\lambda-3)^6$ and geometric multiplicity $m_g(A_i,3)=3$, such that A_i is not similar to A_j for $i \neq j$.

Problem 2 (10+20, symmetric matrices)**.**
(i) Let A and B be two real square matrices of size n which are diagonalizable of $\mathbb{C}$. Suppose we have for their characteristic polynomials: $p_A = p_B$. Prove that A is similar to B over $\mathbb{R}$.
(ii) Which of the following three matrices are similar to each other:

$$\begin{pmatrix} 14 & -3 & 1 \\ -3 & 9 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 9 & -3 & 6 \\ -3 & 11 & -1 \\ 6 & -1 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 7\frac{\sqrt{2}}{\sqrt{5}} & \\ 7\frac{\sqrt{2}}{\sqrt{5}} & 5 & \\ & & 15 \end{pmatrix}$$

Problem 3 (40, higher eigenspaces)**.** Find all real higher eigenspaces of the following matrtices:

$$\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 3 & \\ & & 3 \end{pmatrix}, \begin{pmatrix} -1 & 12 & -8 \\ -8 & 27 & -16 \\ -12 & 36 & -21 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 \\ -2 & 5 & 4 \\ 2 & -2 & -1 \end{pmatrix}.$$

Problem 4 (40, company matrix)**.** Let $a_0, a_1, a_2, \ldots, a_{n-1}$ be real numbers.
(i) Compute the characteristic polynomial of

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & \ddots & & \vdots & \vdots \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}.$$

(ii) Let λ be a complex eigenvalue of A. Show that its geometric multiplicity is 1.
(iii) Is the matrix

$$\begin{pmatrix} & & & 2 \\ 1 & & & 3 \\ & 1 & & -1 \\ & & 1 & -3 \end{pmatrix}$$

diagonlizable over $\mathbb{R}$?
(iv) Find a real matrix A of size 4 with complex eigenvalues $1-i, 1+i, 2, 3$. Given two such matrices, are they similar over $\mathbb{R}$?

Problem 5 (20*, complex and real eigenspaces)**.** Let A be a real square matrix of size n and $\lambda \in \mathrm{Spec}_{\mathbb{R}}(A)$. Then:
(i) $\mathrm{Eig}_{\mathbb{C}}(A,\lambda) = \mathrm{Eig}_{\mathbb{R}}(A,\lambda) + i\mathrm{Eig}_{\mathbb{R}}(A,\lambda)$.
(ii) $\dim_{\mathbb{C}} \mathrm{Eig}_{\mathbb{C}}(A,\lambda) = \dim_{\mathbb{R}} \mathrm{Eig}_{\mathbb{R}}(A,\lambda)$.

Date: Please hand in before the lecture by **11th of December 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.

Problem 6 (10+10*, direct sum of eigenspaces)**.** Let V be a vector space and $\mathrm{W}_1, \mathrm{W}_2, \ldots, \mathrm{W}_r$ be subspaces. We call the sum

$$\mathrm{W}_1 + \mathrm{W}_2 + \mathrm{W}_3 + \ldots + \mathrm{W}_r$$

a *direct sum* if the zero vector **cannot** be obtained as a sum $w_1 + w_2 + \ldots + w_r$ with $w_i \in \mathrm{W}_i$ with some summand being non-zero. Let A be a square matrix of size n and $\lambda_1, \ldots, \lambda_r$ be distinct real eigenvalues of A.

(i) Prove that

$$\mathrm{Eig}(\mathrm{A}, \lambda_1) + \mathrm{Eig}(\mathrm{A}, \lambda_1) + \ldots + \mathrm{Eig}(\mathrm{A}, \lambda_r)$$

is direct.

(ii) We call $\mathrm{Eig}^i(\mathrm{A}, \lambda) := ((\mathrm{A} - \lambda \mathrm{I}_n)^i)$ the ith *higher eigenspace* of A corresponding to λ. Given any tuple of positive integers $\nu_1, \nu_2, \ldots, \nu_r$, prove that

$$\mathrm{Eig}^{\nu_1}(\mathrm{A}, \lambda_1) + \mathrm{Eig}^{\nu_2}(\mathrm{A}, \lambda_1) + \ldots + \mathrm{Eig}^{\nu_r}(\mathrm{A}, \lambda_r)$$

is direct.

Problem 7 (20*, spectral norm)**.** Compute the spectral norm of

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

see Theorem 242 in the pdf about applications for Chapter 5.