LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 10

PROF. DANIEL SKODLERACK

Problem 1 (20 points, eigenvalues and eigenvectors)**.** Find the eigenvectors and eigenvalues for the following matrix A:

$$\begin{pmatrix} -1 & 1 & 0 & -1 \\ -12 & 4 & 4 & -1 \\ 0 & 1 & -1 & -1 \\ -8 & 1 & 4 & 2 \end{pmatrix}.$$

.

Is the matrix diagonalisable? If so please provide a matrix P which conjugates A to a diagonal matrix.

Problem 2 (20+10 points, diagonalisable matrices)**.**

(i) Find the eigenvalues and the eigenvectors of the matrix A =

$$\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \end{pmatrix}$$

Is the matrix diagonalisable? If so please provide a matrix P which conjugates A to a diagonal matrix.

(ii) Consider the sequence

$$a_{-1} := a_0 := 3,\ a_1 := 7,\ a_{n+1} := 3a_n - 2a_{n-1},\ n \geqslant 1.$$

Find an explicit formula for a_n and prove your formula. Hint: Consider the matrix

$$\begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix}$$

Problem 3 (30, complex numbers)**.**

(i) Let V be a real vector space and $f : \mathrm{V} \to \mathrm{V}$ be an $\mathbb{R}$-linear map such that $f \circ f = -\mathrm{id}_{\mathrm{V}}$. $(f(f(v)) = -v\ )$ We define

$$(a + bi) \odot v := av + bf(v).$$

Prove that $(\mathrm{V}, +, \odot)$ is a complex vector space.

(ii) $\mathrm{V} = \mathbb{R}^2$. We define a complex scalar multiplication $\odot$ via

$$(a + bi) \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} ax_1 + b(x_1 - 2x_2) \\ ax_2 + b(x_1 - x_2) \end{pmatrix}$$

Prove that $(\mathrm{V}, +, \odot)$ is a complex vector space.

(iii) $\mathbb{C} = \mathbb{R}^2$ is a complex vector space using complex multiplication as scalar multiplication. Find the corresponding $\mathbb{R}$-linear map f from (i) which describes this complex scalar multiplication.

Problem 4 (10+10+10* points, similar matrices)**.**

(i) Prove that the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is similar to its transpose.

Date: Please hand in before the lecture by **4th of December 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.

(ii) Let A be an upper triangular matrix where all diagonal entries are pairwise different. Prove that A is similar to its transpose.

(iii) (*) Let A and B be real square matrices of size n. Suppose A and B are similar over $\mathbb{C}$, i.e. there exists an invertible matrix $\mathrm{C} \in \mathbb{C}^{n\times n}$ such that $\mathrm{C}^{-1}\mathrm{A}\mathrm{C} = \mathrm{B}$. Prove that A and B are similar over $\mathbb{R}$, i.e. there exists an invertible matrix $\mathrm{F} \in \mathbb{R}^{n\times n}$ such that $\mathrm{F}^{-1}\mathrm{A}\mathrm{F} = \mathrm{B}$.
Hint: Consider for a certain matrix $\mathrm{C} = \mathrm{C}_1 + i\mathrm{C}_2,\ \mathrm{C}_1, \mathrm{C}_2 \in \mathbb{R}^{n\times n}$*, the polynomial function* $p(\lambda) = \det(\mathrm{C}_1 + \lambda \mathrm{C}_2),\ \lambda \in \mathbb{C}$.