LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 9

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Problem 1 (10+10, transition matrix). (i) Consider the bases

$$\mathrm{B} := \left\{(1,1,0)^T, (1,1,-1)^T, (1,2,1)^T\right\} \text{ and } \mathrm{C} := \left\{(1,0,1)^T, (3,1,1)^T, (1,-1,1)^T\right\}$$

of $\mathbb{R}$.

(a) Compute the coordinate vector of $(4,3,1)^T$ with respect to B.

(b) Compute the transition matrix $\mathrm{P}_{\mathrm{B}\to\mathrm{C}}$.

(ii) Let $\mathrm{V} := \{P \in \mathrm{Poly}(\mathbb{R})| \ \deg(P) \leqslant 3\}$ (the set of polynomial functions of degree at most 3). Let W be the span of $\mathrm{B} := \{1-x,\ 1+x-x^2,\ x^3-x+2\}$. For which $\alpha \in \mathbb{R}$ is p_α, defined via $p_\alpha(x) = \alpha - 8x + 5x^2 + x^3$, an element of W. Compute the coordinate vector of p_α with respect to B.

Problem 2 (10+10, linear map). (i) We denote by $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Compute a basis of the $\mathbb{R}$-vector space $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ and compute the dimension.

(ii) Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined via

$$f(x,y) := (\sin(xy), \cos(xy) + x, x + y^2)^T,\ \ x, y \in \mathbb{R}.$$

Let R be the derivative of f at $(1,0)$. Note that $\mathrm{R} \in \mathrm{L}(\mathbb{R}^2, \mathbb{R}^3)$. Compute $[\mathrm{R}]_{\mathrm{B}\to\mathrm{C}}$, the transformation matrix of R with respect to

$$\mathrm{B} := \left\{(1,-1)^T, (1,2)^T\right\} \text{ and } \mathrm{C} := \left\{(1,2,1)^T, (1,1,0)^T, (1,-1,1)^T\right\}.$$

Problem 3 (10+10, Geometry of linear transformations). We are given two vectors $v = (1,1,2)^T$, $w = \sqrt{3}(1,0,1)^T \in \mathbb{R}^3$.

(i) Find a linear rotation R by 60 degree such that $\mathrm{R}v = w$. Determine its axis of rotation.

(ii) For which angles $\theta \in [0, \pi]$ there is a linear rotation R such that $\mathrm{R}v = w$.

Date: Please hand in before the lecture by **27th of November 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.