Mock exam for the midterm

Specific Instructions for students:

• The time duration for this exam is 100 **minutes**.

• Computers and calculators are prohibited in the exam.

• Answers can be written in **either Chinese or English**.

★ For problems 3-7, please show details of calculations or deductions. A correct answer with no details can not earn points.

Policy for grading the Multiple choice questions:

For a multiple choice question, denote by C the set of all correct choices, and by A the set of your choices. If $A \nsubseteq C$, get zero points; If $A \subsetneq C$, get partial credits depending on the size of A.

- If $|C| = 4$, get one point for each correct choice when $|A| < 4$, and get full points when $|A| = 4$;
- If $|C| = 3$, get two points for each correct choice when $|A| < 3$, and get full points when $|A| = 3$;
- If $|C| = 2$, get three points when $|A| = 1$, and get full points when $|A| = 2$.

The unlisted remaining case for $|C| = 1$ should be self evident.

Notations and conventions:

• $\mathbb{R}$ is the set of real numbers.

• I denotes an identity matrix of suitable size.

• 0 or $\mathbf{0}$ may denote the number zero, a zero vector, or a zero matrix.

• $\mathbb{M}_{m\times n}$ is the vector space of all $m \times n$ matrices (with real entries).

• For a square matrix $A = [a_{ij}]$, M_{ij} is the minor of entry a_{ij}; C_{ij} is the cofactor of entry a_{ij}; $adj(A)$ is the adjoint (adjunct) matrix of A.

• For a square matrix A, both $det\,(A)$ and $|A|$ denote the determinant of A.

• Give a matrix A, we denote by $null(A)$, $row\,(A)$, $col(A)$ the null space, row space, column space of A respectively. And $nullity\,(A)$ and $r(A)$ denotes the nullity and rank of A.

• For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, we denote by $\mathbf{u} \bullet \mathbf{v}$ the *dot product* of $\mathbf{u}$ and $\mathbf{v}$.

• P_n is the vector space of all polynomials (with real coefficients) with degree no more than n.

1. Multiple choice questions.

a). (5 *points*) Which of the following sets are vector spaces? (   )

(A) $\{(a,b) \in \mathbb{R}^2 : \; b = 2a+3\} \subseteq \mathbb{R}^2$, with the usual " + " and " · " as in $\mathbb{R}^2$.

(B) $\{\mathbf{v} \in \mathbb{R}^3 : \; ||\mathbf{v}|| = 1\} \subseteq \mathbb{R}^3$, with the usual " + " and " · " as in $\mathbb{R}^3$.

(C) {All polynomials in P_2 that are divisible by $x-2$}, with the usual " + " and " · " as in P_2.

(D) The set $\mathbb{R}^2$, with addition and scalar multiplication given by: for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $k \in \mathbb{R}$, $\mathbf{x} + \mathbf{y} := (x_1 + 2y_1, \, x_2 + 3y_2)$, $k\mathbf{x} := (kx_1, kx_2)$.

b). (5 *points*) Determine which of the following statements are true. (   )

(A) If $A \in \mathbb{M}_{n\times n}$ is invertible, then its adjoint $adj(A)$ is also invertible.

(B) Let $E \in \mathbb{M}_{3\times 3}$ be an elementary matrix such that $det(E) = 1$, then E must be the identity matrix in $\mathbb{M}_{3\times 3}$.

(C) Let $V \subsetneq \mathbb{R}^5$ be a subspace, then any set of five vectors in V is linearly dependent.

(D) If $A \in \mathbb{M}_{4\times 7}$, and $dim\,(null(A)) = 3$, then for all $\mathbf{b} \in \mathbb{R}^4$, the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution.

c). (5 *points*) Consider a linearly independent set $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq V$ for some $m \geq 1$, and let $\mathbf{v} \in V$. Which possible values can $dim\,(span\{\mathbf{v}_1 + \mathbf{v}, \cdots, \mathbf{v}_m + \mathbf{v}\})$ take? ( )

(A) $m-1$ (B) m (C) $m+1$ (D) $m+2$

2. Fill in the blanks.

a.) (5 *points*) Let $A = \begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}$. Then $(adj(A))^{-1} =$ ________.

b). (5 *points*) The distance of $P = (1,1,0)$ to the plane given by $2x + y - 3z = 5$ is equal to ________.

c.) (5 *points*) Let $A = (a_{i,j}) \in \mathbb{M}_{n\times n}$ be given such that $a_{i,j} = ij$ for all $i, j = 1, \cdots, n$. Assuming that $n \geq 2$, then $det\,A =$ ________.

3.

a.) (5 *points*) Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, and suppose that $A^2 - AB = I_3$. Find B.

b.) (5 *points*) Compute the following determinant

$$\begin{vmatrix} 1 & -1 & 2 & 2 \\ 2 & 0 & 1 & 0 \\ 3 & 2 & -2 & 1 \\ -4 & 2 & 1 & 1 \end{vmatrix}.$$

4. Let $A \in \mathbb{M}_{4\times 5}$ be the following matrix

$$\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & 3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix}.$$

a). (10 *points*) Compute $r(A), nullity(A)$, and find a basis for $row\,(A), col\,(A)$ and $null(A)$.

b). (5 *points*) Determine whether $\mathbf{u} = [2, 1, 7, -12]^T$ belongs to $col(A)$.

c). (5 *points*) Find the space of all vectors in $\mathbb{R}^4$ that are orthogonal to $col(A)$, i.e. the *orthogonal complement* of $col(A)$ in $\mathbb{R}^4$.

5. Let $\mathbb{M}_{2\times 2}$ denote the vector space of all 2×2 matrices with real entries. Consider the following two subsets of $\mathbb{M}_{2\times 2}$

$$U = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} : x, y, z \in \mathbb{R} \right\}; \qquad W = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

a). (10 *points*) Verify that both U and W are vector subspaces of $\mathbb{M}_{2\times 2}$. And find a basis and compute the dimension for U and W.

b). (10 *points*) Find the dimension and a basis for the subspaces $U + W$ and $U \cap W$.

6. a). (5 *points*) Let $\mathbf{v}_1 = [1, 3, 0, 2]^T$, $\mathbf{v}_2 = [-1, 0, 1, 0]^T$, $\mathbf{v}_3 = [5, 9, -2, 6]^T$ be vectors in $\mathbb{R}^4$. Is it possible to find a set of numbers $\{a_{ij} \mid i, j = 1, 2, 3\}$, such that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent? Here the vectors $\mathbf{w}_i$'s are given by

$$\begin{aligned} \mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + a_{13}\mathbf{v}_3 \\ \mathbf{w}_2 &= a_{21}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + a_{23}\mathbf{v}_3 \\ \mathbf{w}_3 &= a_{31}\mathbf{v}_1 + a_{32}\mathbf{v}_2 + a_{33}\mathbf{v}_3 \end{aligned}$$

Please provide a full explanation of your claim.

b). (5 *points*) Prove or disprove the following statement. Every rank 1 square matrix of size n can be written as $\mathbf{u}\mathbf{v}^T$ for some n dimensional non zero column vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

7. (10 *points*) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ be two linearly independent sets of vectors in $\mathbb{R}^n$ for some integer n such that $\mathbf{v}_i \bullet \mathbf{w}_j = 0$ for all $i = 1, 2, 3$ and $j = 1, 2$. Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2\}$ still linearly independent? Verify your claim.