LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 7

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Problem 1 (10 + 10 + 10*, warm up for vector spaces). (i) Let $(\mathrm{V}, +, \cdot)$ be a vector space. We have seen in the lecture that there is exactly one neutral element in V.

(a) Prove that for all $v \in \mathrm{V}$ there exists exactly one additive inverse. We denote it by $-v$.

(b) Prove that for every $v \in \mathrm{V}$ and $\lambda \in \mathbb{R}$ we have

$$-(-v) = v,\ (-1)v = -v,\ (-\lambda)(-v) = \lambda v,\ 0_{\mathbb{R}} v = 0_{\mathrm{V}}.$$

(ii) Show that the "strange" example, see Example 153(e), is a vector space.

(iii) (*) Think about why the "strange" example is in fact not so strange.

Problem 2 (10 + 10* + 10 + 10, subspaces). (i) Consider the following subsets $\mathscr{S}$ of $\mathrm{V} := \mathrm{Map}(\mathbb{N}, \mathbb{R})$. Decide whether or not $\mathscr{S}$ is a subspace of V. Prove your answer. If $\mathscr{S}$ is not a subspace then find a maximal subspace in $\mathscr{S}$ and compute the subspace generated by $\mathscr{S}$.

(a) $l^1 := \{(a_n)_{\mathbb{N}} |\ \sum_{n=1}^{\infty} |a_n| < \infty\}$

(b) (*) $l^p := \{(a_n)_{\mathbb{N}} |\ \sum_{n=1}^{\infty} |a_n|^p < \infty\},\ p \in]1, \infty[$

(c) $\mathrm{W} := \{(a_n)_{\mathbb{N}} | \sup\{|a_n|^n |\ n \in \mathbb{N}\} < \infty\}$

(ii) Find all subspaces W of $\mathbb{R}^3$ containing $(1, 1, 2)$.

Problem 3 (20, Linear independence). Which of those families of vectors are linearly independent? Compute the dimension of the subspace generated by those vectors.

(i) $(1, -2, 3, 1), (1, 2, 1, 1), (1, 0, 0, 1)$ in $\mathbb{R}^4$

(ii) $(1, 2, 1, -1), (2, 1, 0, 0), (1, 2, 1, 1), (0, -1.5, -1, -3)$ in $\mathbb{R}^4$

(iii) In $\mathrm{Map}(\mathbb{R}, \mathbb{R})$: $1, \sin(x), \sin(2x), \sin(3x)$

(iv) In $\mathrm{Map}(\mathbb{R}, \mathbb{R})$: $1, \sin(x), \sin(2x), \sin(3x), \sin^3(x)$

Problem 4 (30, Basis of a vector space). Find a basis for the following three vector spaces.

(i) $\mathrm{W}_1 := \mathrm{Span}\{(1, 1, 0, 1)^T, (3, 1, 0, 1)^T, (-1, 1, 0, 1)^T, (0, 1, 1, 0)^T\}$

(ii) $\mathrm{W}_2 := \{x \in \mathbb{R}^3 |\ x_1 + x_2 - x_3 = 0\}$

(iii) $\mathrm{W}_3 := \{x \in \mathbb{R}^4 |\ 2x_1 + x_4 = 3x_3 + 5x_1 = 0\}$

Problem 5 (10*+10*+10*, base extension theorem). (i) (*) Let V be a vector space and U, W be subspaces. Show that their intersection and their sum

$$\mathrm{U} \cap \mathrm{W},\ \mathrm{U} + \mathrm{W} := \{w + u |\ w \in \mathrm{W},\ u \in \mathrm{U}\}$$

are subspaces of V.

(ii) (*) Prove the base extension theorem, see Theorem 183.

(iii) (*) Prove Proposition 190 in the notes: Let V be a finite dimensional vector space and W_1, W_2 be subspaces of V. Show that

$$\dim(\mathrm{W}_1 + \mathrm{W}_2) = \dim(\mathrm{W}_1) + \dim(\mathrm{W}_2) - \dim(\mathrm{W}_1 \cap \mathrm{W}_2).$$

Date: Please hand in before the lecture by **13th of November 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.