LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 6

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Problem 1 (10+10 points, hyperplanes)**.** Consider the following two pairs of hyperplanes in $X = \mathbb{R}^3$. If they intersect then compute the angle between them. If they don't intersect then compute their distance.

(i) $\mathrm{H}_1 := \{(x_1, x_2, x_3) \in \mathrm{X} \mid \ x_1 + 2x_2 + x_3 = 1\}$ and
$\mathrm{H}_2 := \{\mathrm{P} + \vec{v} \mid \ \vec{v} \in \mathbb{R}\overrightarrow{(1, 1, -3)} + \mathbb{R}\overrightarrow{(1, -1, 1)}\}, \ \mathrm{P} = (0, 1, 0)$

(ii) H_1 and $\mathrm{H}_3 := \{\mathrm{P} + \vec{v} \mid \ \vec{v} \in \mathbb{R}\overrightarrow{(1, 1, 0)} + \mathbb{R}\overrightarrow{(1, 1, 1)}\}$

Problem 2 (20 points, distance between two lines)**.** Consider the lines:

$$\mathrm{L}_1 := (1, 1, 1, 1) + \mathbb{R}\overrightarrow{(2, -1, 2, 1)}, \ \mathrm{L}_2 := (1, 0, 0, 0) + \mathbb{R}\overrightarrow{(1, -1, 1, 1)}$$

Find the distance between those lines and find all pairs of points $\mathrm{A} \in \mathrm{L}_1, \mathrm{B} \in \mathrm{L}_2$ which are closest to each other, i.e. such that $d(\mathrm{A}, \mathrm{B}) = \mathrm{dist}(\mathrm{L}_1, \mathrm{L}_2)$.

Problem 3 (10 + 10 + 10* + 20*,cross product and generalisation)**.**

(i) Compute the following cross products

$$(1, 3, 5) \times (-1, 2, 6), \ (-2, 1, -1) \times (1, 1, -1)$$

(ii) Find a normal for the hyperplane through the points $(1, 1, 3), (1, -1, 1), (1, 1, 1)$ whose length is the area of the triangle with those points as vertexes.

(iii) (*) Try to generalise the cross product to higher dimensions, i.e. define a suitable vector product $\times(v^{(1)}, v^{(2)}, \ldots, v^{(n-1)}) \in \mathbb{R}^n$ for vectors $v^{(1)}, \ldots, v^{(n-1)} \in \mathbb{R}^n$. Give a formula and describe the length and the direction.

(iv) (*) Most ambitious: Give a proof for your assertions in (iii).

Problem 4 (20 points, lines and planes)**.** Describe the following sets with (a) a vector form, (b) parametric equations and (c) as an intersection of hyperplanes.

(i) The line through the points $(1, 0, 0, 1)$ and $(2, 2, 0, 1)$.

(ii) The plane throught the points $(1, 1, 1, 0), (1, 0, 1, 0)$ and $(1, 2, 1, 2)$.

Date: Please hand in before the lecture by **6th of November 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.