LINEAR ALGEBRA 1 (FALL 2024)
PROBLEM SHEET 4

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Problem 1 (5+5+10 points, computing determinants)**.** Compute the determiant for the following matrices.

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & -4 & -2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & -1 \\ 1 & 3 & 1 & 2 \\ 1 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -3 & 1 & 5 \\ 1 & 2 & \frac{1}{2} & 6 & 1 \\ 0 & 1 & 2 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & b \end{pmatrix}.$$

For which $b \in \mathbb{R}$ is the last matrix invertible?

Problem 2 (10+10, computing determinants)**.** Compute the following determinants:

(i)

$$\begin{vmatrix} -2 & 1 & 3 & 1 & 1 & -2 & 4 & 1 & 1 \\ 1 & 3 & 5 & 1 & -1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 & 5 & -1 & 1 \\ 2 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 \\ 1 & -1 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ -1 & 1 & 2 & 1 & 1 & -1 & 1 & 3 & 2 \\ 0 & 1 & -2 & 1 & 3 & 2 & 1 & 1 & 1 \end{vmatrix}.$$

(ii)

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{vmatrix}.$$

for real numbers λ_i, $i = 1, \dots, m$ and $m \geqslant 2$.. (Hint: Perform a row operation with the last two rows.)

Problem 3 (10 points, determinant and (E2))**.** Prove Proposition 73, i.e. that swapping two rows does change the sign of the determinant. You are only allowed to use the definition of the determinant given in Definition 64.

Problem 4 (10*+5+5 points, (E3))**.** Let A be a real $m \times n$-matrix.

(i) Show that there is a matrix $\mathrm{E} \in \mathbb{R}^{m \times m}$, which is a product of addition matrices, see (E3), and there exists a diagonal matrix $\mathrm{D} \in \mathbb{R}^{m \times m}$ with non-zero determinant such that

$$\mathrm{EA} = \mathrm{DR}$$

where R is the reduced row echelon form of A.

(ii) Show that the diagonal matrix $\mathrm{diag}(\lambda, \frac{1}{\lambda}) \in \mathbb{R}^{2 \times 2}$ with non-zero $\lambda \in \mathbb{R}$ is a product of addition matrices.

(iii) Suppose that $n = m$ and A has determinant 1. Show that A is a product of addition matrices.

Date: Please hand in before the lecture by **23rd of October 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. * questions give extra points.