Chapter VIII

Linear transformations

VIII 1. 1st Definitions

We have already defined linear maps in Def 197.
This chapter is devoted to study those map thoroughly.

Motivation: 1) Most vector spaces are not n-spaces, because

- they are infinite dimensional or
- they have no canonical choice of a basis.

2) The matrix $A \in \mathbb{R}^{m\times n}$ defining $T_A : \mathbb{R}^{n\times 1} \longrightarrow \mathbb{R}^{m\times 1}$, $T_A(v) = Av$ may not be the "nicest" matrix to describe the map.
So it is better to have a coordinate independent concept of linear map.

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Def 3.14: Let V, W be vector spaces.

1) A map $f: V \longrightarrow W$ is called linear if

- $\forall \lambda \in \mathbb{R} \, \forall v \in V: f(\lambda v) = \lambda f(v)$ and (homogeneous)
- $\forall v_1, v_2 \in V: f(v_1 + v_2) = f(v_1) + f(v_2)$ (additive)

A linear map is also called a linear transformation or homomorphism

2) A linear transformation $f: V \to W$ is called a monomorphism if f is injective (one-to-one)
(Recall that means
For $v_1 \neq v_2$ in V we have $f(v_1) \neq f(v_2)$.)

3) A linear transformation is called epimorphism if f is surjective (onto)
(Recall that means
$\forall w \in W \, \exists v \in V: f(v) = w$)

4) A linear transformation $f:V\to W$ is called isomorphism if f is bijective, i.e. f is surjective and injective.

Example 315: (see also Example 321)

1) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=2x$

$$f(x_1+x_2)=2(x_1+x_2)=2x_1+2x_2=f(x_1)+f(x_2)$$

$$f(\lambda x)=2(\lambda x)=(2\lambda x)=\lambda(2x)=\lambda f(x).$$

So f is additive and homogeneous

Thus f is linear.

2) $f:\mathbb{R}\to\mathbb{R}$ $\quad f(x)=x^2.$

Then $f(1+1)=f(2)=4$

$f(1)+f(1)=1^2+1^2=2\neq 4.$

So f is not addive $\Rightarrow$ f is not linear.

3) $f:\mathbb{R}^{3\times 1}\longrightarrow\mathbb{R}^{2\times 1}$

$$f\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right)=\begin{pmatrix}x_1-2x_2\\x_3-x_1+x_2\end{pmatrix}$$

$$f\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\right)=f\left(\begin{pmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{pmatrix}\right)=\longrightarrow$$

$$\begin{pmatrix} (x_1+y_1) - 2(x_2+y_2) \\ (x_3+y_3) - (x_1+y_1) + (x_2+y_2) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - 2x_2 \\ x_3 - x_1 + x_2 \end{pmatrix} + \begin{pmatrix} y_1 - 2y_2 \\ y_3 - y_1 + y_2 \end{pmatrix}$$

$$= f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) + f\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right)$$

so f is additive.

$$f\left(\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \lambda x_1 - 2(\lambda x_2) \\ \lambda x_3 - \lambda x_1 + \lambda x_2 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} x_1 - 2x_2 \\ x_3 - x_1 + x_2 \end{pmatrix} = \lambda f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right).$$

So f is homogeneous.
Thus f is linear.

4) $f : C^1(\mathbb{R}) \longrightarrow C(\mathbb{R})$

$f(\varphi) := \varphi'$ derivation

$\cdot\ f(\varphi_1 + \varphi_2) = (\varphi_1 + \varphi_2)' = \varphi_1' + \varphi_2'$

$= f(\varphi_1) + f(\varphi_2)$

- $f(\lambda\varphi) = (\lambda\varphi)' = \lambda\varphi' = \lambda f(\varphi)$ for $\lambda \in \mathbb{R}$, $\varphi \in C^1(\mathbb{R})$.

5) $f: C(\mathbb{R}) \to C^1(\mathbb{R})$

$$f(\varphi) := \underbrace{\int_0^x \varphi(t)\,dt}_{x \longmapsto \int_0^x \varphi(t)\,dt}$$

$$\begin{aligned}(f(\varphi+\psi))(x) &= \int_0^x \varphi(t)+\psi(t)\,dt \\ &= \int_0^x \varphi(t)\,dt + \int_0^x \psi(t)\,dt \\ &= (f(\varphi))(x) + (f(\psi))(x) \\ &\underset{\substack{\uparrow \\ \text{addition} \\ \text{in } C^1(\mathbb{R})}}{=} (f(\varphi)+f(\psi))(x).\end{aligned}$$

$f(\lambda\varphi) = \lambda f(\varphi)$. So f is linear.

<u>Example 316:</u> 1) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 2x$

injective? $x_1 \neq x_2 \Rightarrow f(x_1) = 2x_1$

$$f(x_2) = 2\overset{\neq}{x_2}$$

$\Rightarrow f$ is injective.

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Better way to show injectivity:

$\forall_{x_1, x_2}: (f(x_1) = f(x_2) \Longrightarrow x_1 = x_2)$

Take $x_1, x_2 \in \mathbb{R}$. Suppose $f(x_1) = f(x_2)$

$\Longrightarrow \quad 2x_1 = 2x_2 \Rightarrow 2(x_1 - x_2) = 0$

$\underset{\cdot \frac{1}{2}}{\overset{}{\underset{\uparrow}{\Longrightarrow}}} \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$

So f is injective.

surjective? Take $y \in \mathbb{R}$.

We need to find $x \in \mathbb{R}$ such that $f(x) = y$.

$y = f(x) = 2x \Longrightarrow x = \frac{y}{2}.$

Take $x := \frac{y}{2}.$

Then $f(x) = 2\left(\frac{y}{2}\right) = y.$

So f is surjective.

bijective? yes, because injective and surjective.

2) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^2$

injective? No! $f(1) = 1 = f(-1)$.

surjective? No! $-1 = f(x) = x^2$ has no solution in $\mathbb{R}$.

3) $f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$f: \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{2\times 1}$.

injective? No! $f\left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = f\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$

surjective? Take $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^{2\times 1}$

Then $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -y_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} - y_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= f\left(\begin{pmatrix} -y_1 \\ -y_1 \\ y_2 \end{pmatrix}\right)$.

So f is surjective.

bijective? No! because <u>not</u> injective.

4) $f : C^1(\mathbb{R}) \to C(\mathbb{R})$ $\quad f(\varphi) = \varphi'$

injective? No! $f(1) = 0 = f(0)$

constant maps

surjective? Yes! Take $\varphi \in C(\mathbb{R})$

$$f\left(\int_0^x \varphi(t)\,dt\right) = \left(\int_0^x \varphi(t)\,dt\right)'$$

$$= \varphi$$

5) Study $f : C(\mathbb{R}) \longrightarrow C^1(\mathbb{R})$

$$f(\varphi) = \int_0^x \varphi(t)\,dt.$$

wrt. injectivity and surjectivity.

Prop. 317: 1) For $A \in \mathbb{R}^{m\times n}$ the map $T_A : \mathbb{R}^{n\times 1} \to \mathbb{R}^{m\times 1}$ $\quad T_A(v) := Av$ is linear

2) Let $f : \mathbb{R}^{n\times 1} \longrightarrow \mathbb{R}^{m\times 1}$ be linear. Then $\exists\, A \in \mathbb{R}^{m\times n} : f = T_A$.

Proof: 1) $T_A(v_1+v_2) = A(v_1+v_2)$

$= Av_1 + Av_2 = T_A(v_1) + T_A(v_2)$

$T_A(\lambda v) = A(\lambda v) = \lambda(Av) =$

$= \lambda T_A(v)$

2) We are given a linear transf. $f: \mathbb{R}^{n\times 1} \longrightarrow \mathbb{R}^{m\times 1}$.

Consider $\begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} = f(\vec{e}_i)$

Put $A := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

Claim $f = T_A$.

Proof: We know that f and T_A are linear.

Exercise: We only need to show that f and T_A coincide on generating set.

Take $S = \{\vec{e}_1, \dots, \vec{e}_n\}$

$$f(\vec{e}_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} = A\vec{e}_i = T_A(\vec{e}_i) \checkmark$$

for $i=1,\dots,n$.

S generates $\mathbb{R}^{n\times 1}$. Thus $f = T_A$ □ (Claim)

□ (Prop 317)

Thus for $\mathbb{R}^n$ we don't get something new, but it is still helpful, because A could be not very beautiful.

Example 318: (a) $f : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{2\times 1}$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$$

Find A!

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So $f = T_{\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}}$.

(b) $f : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{2\times 1}$
orthogonal projection onto $\mathbb{R}\begin{pmatrix}1\\1\end{pmatrix}$.
Find A, s.t. $f = T_A$.

$$f\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}, \quad f\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix}.$$
(because $\begin{pmatrix}1\\-1\end{pmatrix} \perp \begin{pmatrix}1\\1\end{pmatrix}$)

Want $A \in \mathbb{R}^{2\times 2}$ such that

$$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \text{ and } A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \text{ i.e.}$$

$$A\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \begin{pmatrix}1&0\\1&0\end{pmatrix}.$$

Invert $\begin{pmatrix}1&1\\1&-1\end{pmatrix}$: $\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1} = \frac{1}{-2}\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

So $A = \begin{pmatrix}1&0\\1&0\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \underline{\underline{\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}}}$.

So $f = T_{\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&+\frac{1}{2}\end{pmatrix}}$.

Check: $\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$ ✓

-"- $\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ ✓

Def. 319: 1) Let V, W be vector spaces
$\mathrm{Hom}(V,W) := \{ f : V \to W \mid f \text{ linear} \}$

2) Take $f \in \mathrm{Hom}(V,W)$.

$\ker(f) := \{ v \in V \mid f(v) = 0 \}$
kernel of f

$\mathrm{range}(f) := \{ f(v) \mid v \in V \}$
range of f
(also image of f)

Theorem 320: Let $f \in \mathrm{Hom}(V,W)$.

1) $\ker(f)$ and $\mathrm{range}(f)$ are subspaces.

2) f injective $\iff$ $\ker(f) = \{0\}$

3) f surjective $\iff$ $\exists\, S \subseteq W$:
S generates W and $S \subseteq \mathrm{range}(f)$.

We write $\mathrm{nullity}(f) := \dim \ker(f)$
$\mathrm{rank}(f) = \mathrm{rk}(f) := \dim \mathrm{range}(f)$

Proof: 1) Just for ker(f).

To show $\ker(f) \leq V$.

$0 \in \ker(f)$: $f(0) = f(0+0) \overset{f \text{ linear}}{=} f(0)+f(0)$

$$\xRightarrow[-f(0)]{} \quad 0 = f(0). \Rightarrow 0_V \in \ker(f)$$

closed under $+$: $v_1, v_2 \in \ker(f)$.

$$f(v_1+v_2) = f(v_1) + f(v_2) = 0_V + 0_V = 0_V$$

$$\Rightarrow v_1 + v_2 \in \ker(f)$$

closed under $\cdot : \mathbb{R} \times V \to V$: $\lambda \in \mathbb{R}, v \in \ker(f)$

$$f(\lambda v) = \lambda f(v) = \lambda \cdot 0_V = 0_V.$$

$$\Rightarrow \lambda v \in \ker(f).$$

Subspace criterion $\Rightarrow \ker(f) \leq V$.

2) "$\Rightarrow$" For $v \in \ker(f)$ we have

$$f(v) = 0_W = f(0_V).$$

f injective $\Rightarrow v = 0_V$

Thus $\ker(f) = \{0_V\}$.

"$\Leftarrow$" We have $\ker(f) = \{0_V\}$

To show: f is injective.

$$f(v_1) = f(v_2) \Rightarrow \underbrace{f(v_1) - f(v_2)}_{\overset{f \text{ linear}}{=} f(v_1 - v_2)} = 0_W$$

$$\Rightarrow v_1 - v_2 \in \ker(f) = \{0_V\}$$

$$\Rightarrow v_1 - v_2 = 0_V \Rightarrow v_1 = v_2$$

3) "$\Rightarrow$" f surjective $\Rightarrow$ range$(f) = W$

Take S to be W.

"$\Leftarrow$" Suppose $\exists S$ generating W: range$(f) \supseteq S$.

$$\underset{\substack{\uparrow \\ \text{subspace} \\ \text{of } W}}{\Rightarrow} \text{range}(f) \supseteq \text{span}(S) = W.$$

$\Rightarrow$ f is surjective $\square$

(for 315)

Example 321: Let $f: \mathbb{R}^{2\times 1} \to \mathbb{R}^{3\times 1}$ be linear and

$$f\left(\begin{pmatrix}1\\2\end{pmatrix}\right)=\begin{pmatrix}1\\1\\3\end{pmatrix} \text{ and } f\left(\begin{pmatrix}2\\-1\end{pmatrix}\right)=\begin{pmatrix}1\\-1\\2\end{pmatrix}.$$

What is $f\left(\begin{pmatrix}5\\1\end{pmatrix}\right)$?

$$\begin{pmatrix}5\\1\end{pmatrix} = \lambda_1\begin{pmatrix}1\\2\end{pmatrix} + \lambda_2\begin{pmatrix}2\\-1\end{pmatrix} = \frac{7}{5}\begin{pmatrix}1\\2\end{pmatrix} + \frac{9}{5}\begin{pmatrix}2\\-1\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1 & 2 & 5\\ 2 & -1 & 1\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}1 & 2 & 5\\ 0 & -5 & -9\end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c}1 & 0 & 7/5\\ 0 & 1 & 9/5\end{array}\right) \qquad \begin{array}{l}\lambda_1 = \frac{7}{5}\\ \lambda_2 = \frac{9}{5}\end{array}$$

Then $f\left(\begin{pmatrix}5\\1\end{pmatrix}\right) = f\left(\frac{7}{5}\begin{pmatrix}1\\2\end{pmatrix} + \frac{9}{5}\begin{pmatrix}2\\-1\end{pmatrix}\right)$

$$\underset{\uparrow}{=} f\left(\frac{7}{5}\begin{pmatrix}1\\2\end{pmatrix}\right) + f\left(\frac{9}{5}\begin{pmatrix}2\\-1\end{pmatrix}\right)$$

linear

$$\overset{\downarrow}{=} \frac{7}{5} f\left(\begin{pmatrix}1\\2\end{pmatrix}\right) + \frac{9}{5} f\left(\begin{pmatrix}2\\-1\end{pmatrix}\right).$$

$$= \frac{7}{5}\begin{pmatrix}1\\1\\3\end{pmatrix} + \frac{9}{5}\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}16/5\\ -2/5\\ 39/5\end{pmatrix}$$

Example 322:

(a) $f:\mathbb{R}\longrightarrow\mathbb{R}^2$ $\quad f(x)=\begin{pmatrix}x\\2x\end{pmatrix}$.

f is linear, because $f=T_{\begin{pmatrix}1\\2\end{pmatrix}}$.

injective? $\ker(f)=\{x\in\mathbb{R}\mid f(x)=\begin{pmatrix}0\\0\end{pmatrix}\}$

$$=\{x\in\mathbb{R}\mid \begin{pmatrix}x\\2x\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\}=\{0_{\mathbb{R}}\}$$

So f is injective by Thm 320

surjective? No! $\begin{pmatrix}1\\0\end{pmatrix}\notin$ range(f).

(b) $f:\mathbb{R}^{3\times1}\longrightarrow\mathbb{R}^{3\times1}$

$$f\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right)=\begin{pmatrix}x_1+x_2+2x_3\\-x_1+x_2\\x_1-x_2\end{pmatrix}$$

$f=T_A$ for $A=\begin{pmatrix}1&1&2\\-1&1&0\\1&-1&0\end{pmatrix}$.

so is linear

$\ker(f) = \text{null}(A) = \mathbb{R}\begin{pmatrix}1\\1\\-1\end{pmatrix}$

So f is <u>not</u> injective.

<u>surjective?</u>

$\text{range}(f) = \text{col}(A)$ has dimension

(by dim theorem) $3 - \text{nullity}(A) = 3-1 = 2 \neq 3$

So f is not surjective.

(c) $f: \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}^{\mathbb{N}}$

$(a_n)_{\mathbb{N}} \longmapsto (a_{n+1})_{\mathbb{N}}$

f is linear (Exercise)

f is not injective: $f((1,0,0,\dots)) = (0,0,\dots)$

f is surjective: Take $(b_n)_{\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$

Put $\underset{\uparrow}{a_1} = 0$, $a_n := b_{n-1}$ for $n>1$.

$\Rightarrow f((a_n)) = (b_n)_{\mathbb{N}}$.

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(d) Find $f: V \to V$ linear which is injective, but not surjective.

—— End of Lecture 10th of Jan 2024

Theorem 323: Let V, W be finite dimensional vector spaces and f linear and $\dim V = \dim W$.

T.a.e:

1° f is bijective (an isomorphism)

2° f is surjective

3° f is injective.

Proof: 1° ⇒ 2° ✓ and 1° ⇒ 3° ✓ by definition of bijectiveness.

3° ⇒ 1° By the dimension theorem (204) we have

(*) $\text{rk}(f) + \text{nullity}(f) = \dim V$

f injective $\overset{320}{\Longrightarrow}$ $\ker(f) = \{0_V\}$

$\Longrightarrow$ nullity$(f) = 0$

$\overset{(*)}{\Longrightarrow}$ $\operatorname{rk}(f) = \dim V = \dim W$

$\Longrightarrow$ range$(f) = W$ $\Longrightarrow$ f is surjective

$\underset{\substack{\uparrow \\ f \text{ injective}}}{\Longrightarrow}$ f is an isomorphism.

$2^\circ \Longrightarrow 1^\circ$ f is surjective

$\Longrightarrow$ range$(f) = W$ $\Longrightarrow \operatorname{rk}(f) = \dim W$

$\underset{(*)}{\Longrightarrow}$ nullity$(f) = 0$

$\Longrightarrow$ $\ker(f) = \{0_V\}$ $\overset{320}{\Longrightarrow}$ f is injective

$\underset{\substack{\uparrow \\ \text{surjective}}}{\Longrightarrow}$ f is an isomorphism. □

Example 324: (a)

$f: \mathbb{R}^{2\times1} \longrightarrow \mathbb{R}^{2\times1}$ $f\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = \begin{pmatrix}x_1+x_2\\x_1-x_2\end{pmatrix}$

$$\ker(f) = \operatorname{null}\left(\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right) = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

$\overset{320}{\Rightarrow}$ f is injective

$\overset{323}{\Rightarrow}$ f is an isomorphism.

Showing that the kernel is zero is sometimes easier than showing surjectivity.

(b) $$f\left(\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right) := \begin{pmatrix}2x_2 - x_1\\ x_1 + x_2 + 3x_3\\ x_1 + 2x_3\end{pmatrix}, \; f : \mathbb{R}^{3\times1} \to \mathbb{R}^{3\times1}$$

so $$f = T_A, \quad A = \begin{pmatrix}-1 & 2 & 0\\ 1 & 1 & 3\\ 1 & 0 & 2\end{pmatrix}$$

$$\begin{aligned}\ker(f) &= \left\{ v \in \mathbb{R}^{3\times1} \,\middle|\, f(v) = \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}\\ &= \operatorname{null}(A)\\ &= \mathbb{R}\begin{pmatrix}-2\\-1\\1\end{pmatrix} \neq \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}\end{aligned}$$

$\Rightarrow$ f is not injective.

Thm 323 $\Rightarrow$ f is not surjective.

VIII 2. Composition of maps

Def 325: (Composition of maps)
Given two maps
$f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$
we call the following map the composition of g with f:

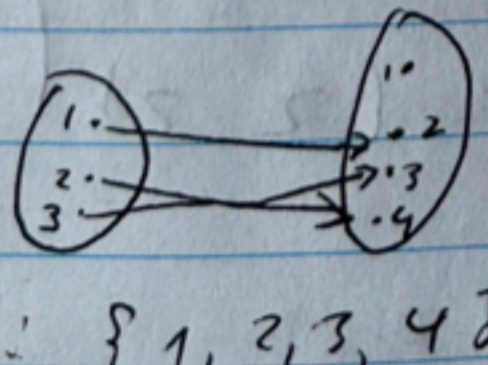
$$(g\circ f)(x) := g(f(x))$$

$$g\circ f : X \longrightarrow Z.$$

(Read: $g\circ f$ "g circle f")

Example 326: (a) $f: \{1,2,3\} \longrightarrow \{1,2,3,4\}$

$$f(i) = \begin{cases} 2, & i=1 \\ 4, & i=2 \\ 3, & i=3 \end{cases}$$



$g: \{1,2,3,4\} \longrightarrow \mathbb{R}$

$g(i) = i^2$

$$(g\circ f)(i) = \begin{cases} 4, & i=1 \\ 16, & i=2 \\ 9, & i=3 \end{cases}$$

(b) $f = T_A : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{3\times 1}$

$g = T_{\mathbb{B}} : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}$

$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} \ , \ \mathbb{B} = (1 \ \ 1 \ \ 0)$

(b1) $(g \circ f)\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = g\left(f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)\right)$

$= g\left(A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = g\left(\begin{pmatrix} x_1 + 2x_2 \\ x_1 \\ -x_1 + x_2 \end{pmatrix}\right)$

$= B\begin{pmatrix} x_1 + 2x_2 \\ x_1 \\ -x_1 + x_2 \end{pmatrix} = (x_1 + 2x_2) + x_1$

$= 2(x_1 + x_2) = (2 \ \ 2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= T_{(2 \ 2)}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$

(b2) $B \circ A = (1 \ \ 1 \ \ 0)\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$

$= (2 \ \ 2)$

So $T_B \circ T_A = T_{B \circ A}$ is also linear.

Prop 3.27: Let $f: V \to W$ and $g: W \to U$ be linear transformations. Then $g \circ f : V \longrightarrow U$ is linear.

Proof: $(g \circ f)(\lambda v) = g(f(\lambda v))$

$$\underset{\substack{\uparrow \\ f \text{ homogeneous}}}{=} g(\lambda(f(v))) \underset{\substack{\uparrow \\ g \text{ homog.}}}{=} \lambda\, g(f(v))$$

$$= \lambda\,(g \circ f)(v)$$

$(g \circ f)(v_1 + v_2) = g(f(v_1 + v_2))$

$$\underset{\substack{\uparrow \\ f \text{ additive}}}{=} g(f(v_1) + f(v_2))$$

$$\underset{\substack{\uparrow \\ g \text{ additive}}}{=} g(f(v_1)) + g(f(v_2))$$

$$= (g \circ f)(v_1) + (g \circ f)(v_2) \quad \square$$

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Theorem 328: Let $f: V \to W$ be linear and V, W finite dimensional

⓪ $id_V : V \to V$ $\quad id_V(v) := v$
is linear. (id_V is called the identity map)

① f is injective $\Leftrightarrow \exists\, g \in Hom(W,V) : g \circ f = id_V$
(g is called a "left inverse of f")

② f is surjective $\Leftrightarrow \exists\, h \in Hom(W,V) :$
$f \circ h = id_W$
(h is called a "right inverse of f")

③ f is an isomorphism
$\Leftrightarrow \exists\, g \in Hom(W,V) : g \circ f = id_V$ and
$f \circ g = id_W$

(In this case g is unique and denoted by f^{-1}, the "inverse of f")

Lemma 329: (Linear extension)

Let V and W be vector spaces and $\{v_1, \ldots, v_n\}$ be a basis of V and $w_1, \ldots, w_n \in W$.
Then $\exists!\ f \in \mathrm{Hom}(V, W)$ such that $f(v_i) = w_i$, $i = 1, \ldots, n$.

Proof: For $v \overset{(*)}{=} \sum_{i=1}^{n} \lambda_i v_i$, $\lambda_i \in \mathbb{R}$

put $f(v) := \sum \lambda_i w_i$.

f is well-def., because $(*)$ is unique.

f is linear:

- $f(\sum \lambda_i v_i + \sum \mu_i v_i) = f(\sum (\lambda_i + \mu_i) v_i)$

$$\underset{\overset{\uparrow}{\text{def}}}{=} \sum (\lambda_i + \mu_i) w_i = \sum \lambda_i w_i + \sum \mu_i w_i$$

$$= f(\sum \lambda_i v_i) + f(\sum \mu_i v_i)$$

so f is additive

- f is homogeneous (exercise) □

Proof of Theorem 3.28:

⓪ exercise.

① "$\Rightarrow$" Suppose f is injective.
Take a basis $\{v_1, \dots, v_n\}$ of V.
Then $\{f(v_1), \dots, f(v_n)\}$ is linear independent.

Pf: $\sum \lambda_i f(v_i) = 0$

$$\Rightarrow f(\sum \lambda_i v_i) = 0 \underset{\substack{\uparrow \\ f \text{ inj}}}{\Rightarrow} \sum \lambda_i v_i = 0$$

$$\underset{\substack{\uparrow \\ \text{basis}}}{\Rightarrow} \lambda_1 = \lambda_2 = \dots = \lambda_n = 0. \quad \square$$

Extend to a basis $\{f(v_1), \dots, f(v_n), w_{n+1}, \dots, w_m\}$

Lem. 3.29 $\Rightarrow \exists g \in \mathrm{Hom}(W, V)$:
$g(f(v_i)) = v_i$, $i = 1, \dots, n$
and $g(w_j) = 0_V$, $j = n+1, \dots, m$.

"$\Leftarrow$" $\exists g \in \mathrm{Hom}(W, V)$: $g \circ f = id_V$
To show f is injective

$f(v) = 0 \Rightarrow v = id_V(v) = (g \circ f)(v)$
$= g(f(v)) = g(0_W) = 0_V. \Rightarrow ker(f) = \{0_V\}.$

② Exercise.

③ f bijective $\Leftrightarrow$ f inj and f surj.

$\overset{1), 2)}{\Leftrightarrow} \exists g, h \in Hom(W, V) : g \circ f = id_V$
and $f \circ h = id_W$

To show $g = h$.

$$h(w) = (id_V \circ h)(w)$$
$$= g(f(h(w))) = g(id_W(w))$$
$$= g(w). \qquad \square$$

<u>Example 330</u>: (a) $f : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}$, $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 + x_2$

is surjective.

Find right inverse $h : \mathbb{R} \to \mathbb{R}^{2\times 1}$

$\{1\}$ is a basis of $\mathbb{R}$.

$f(\vec{e}_1) = 1.$

Take $\in Hom(\mathbb{R}, \mathbb{R}^{2\times 1})$ o, 1.

$h(1) = \vec{e}_1.$

$h(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}.$, $(f \circ h)(x) = x + 0 = x.$

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(b) $f : \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{3\times 1}$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) := \begin{pmatrix} x_1 + x_2 \\ x_2 - x_1 \\ x_1 \end{pmatrix} = T_A\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$ $\ker(f) = \operatorname{null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$\Rightarrow f$ injective

Find left inverse:

$$f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Take g with $g\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$g\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$g\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(Lemma 3.29)

$g \in \operatorname{Hom}(\mathbb{R}^{3\times 1}, \mathbb{R}^{2\times 1})$

Thus $\exists\ B \in \mathbb{R}^{2\times 3} : g = T_B$.

Which B ?

$$B \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{column alg.}} \left(\begin{array}{ccc} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right) \Big\} B$$

(c) $f : \mathbb{R}^{3\times1} \longrightarrow \mathbb{R}^{3\times1}$

$f = T_A \qquad A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$\ker(f) = \text{null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\overset{328}{\Longrightarrow}$ f is an isomorphism.

$f^{-1} = T_{A^{-1}} \qquad A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 2 & -1 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$

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VIII 3. Matrix of a linear transformation w.r.t. to bases.

We understand $f \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ through a matrix. $f = T_A$

But sometimes A is not beautiful.

$\rightsquigarrow$ change the bases.

Def 3.3.1! Let $f \in \mathrm{Hom}(V, W)$
and
$B = \{v_1, \dots, v_n\}$ be a basis of V
$C = \{w_1, \dots, w_m\}$ — " — W

Matrix of f wrt. B and C
(from B to C):

$$[f]_{B \to C} := [f]_{C \leftarrow B}$$

$$:= \left([f(v_1)]_C, [f(v_2)]_C, \dots, [f(v_n)]_C \right)$$

i.e. we write $f(v_1), \dots, f(v_n)$ wrt. C and store the coordinates.

Example 332:

(a) $f: \mathbb{R}^{2\times 1} \longrightarrow \mathbb{R}^{3\times 1}$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_1 \\ 2x_2 + x_1 \end{pmatrix}$$

$$f = T_A \quad , A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[f]_{\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \to \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}} = [f]_{st \to st}$$

$$= A$$

$$\left(f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1\,\vec{e_1} - \vec{e_2} + \vec{e_3} \right)$$

$$\left[f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)\right]_{st} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} ; \quad \left[f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right]_{st} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

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$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} , \quad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = (-3)\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\text{so } \left[f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right]_C = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\left[f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right]_C = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{So } [f]_{C \leftarrow B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \\ 1 & 1 \end{pmatrix}$$

(b) Other way for $[f]_{C \leftarrow \mathcal{B}}$.

$$[f]_{C \leftarrow \mathcal{B}} = P_{C \leftarrow \mathcal{A}} \, [f]_{\mathcal{A} \leftarrow \mathcal{A}} \, P_{\mathcal{A} \leftarrow \mathcal{B}}$$

$$P_{\mathcal{A} \leftarrow \mathcal{B}} = [id_{\mathbb{R}^2}]_{\mathcal{A} \leftarrow \mathcal{B}}$$

$$= \left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\mathcal{A}}, \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]_{\mathcal{A}} \right)$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$P_{C \leftarrow \mathcal{A}} = \left(P_{\mathcal{A} \leftarrow C}\right)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$[f]_{C \leftarrow \mathcal{B}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -2 & -4 \\ 2 & 6 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \\ 1 & 1 \end{pmatrix}$$

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(c) $P_2 := \mathbb{R}p_0 + \mathbb{R}p_1 + \mathbb{R}p_2$

$P_4 := \mathbb{R}p_0 + \mathbb{R}p_1 + \dots + \mathbb{R}p_4$

$p_i(x) = x^i \quad p_i \in Poly(\mathbb{R})$

$g : P_2 \to P_4$ defined via

$g(p) = q$

with $q(x) := p(x^2+1), p \in P_2$.

(c1) g is linear

(c2) $[g]_{\{p_0, p_1, p_2\} \to \{p_0, p_1, \dots, p_4\}}$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Given $V \xrightarrow{f} W \xrightarrow{g} U$

(finite dim vector spaces)

and bases B, C, D.

Then

$$\begin{tikzcd} v \arrow[d] & V \arrow[r, "f"] \arrow[d] & W \arrow[r, "g"] \arrow[d] & U \arrow[d] \\ {[v]_B} & \mathbb{R}^{n\times 1} \arrow[r, "{[f]_{B\to C}}"'] & \mathbb{R}^{m\times 1} \arrow[r, "{[g]_{C\to D}}"'] & \mathbb{R}^{l\times 1} \end{tikzcd}$$

$$[g\circ f]_{D\leftarrow B} = [g]_{D\leftarrow C} \circ [f]_{C\leftarrow B}$$

Ex: $\mathbb{R}^{2\times 1} \xrightarrow{f} P_2 \xrightarrow{g} P_4$

$$f\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \underbrace{(t_1+t_2)}_{\text{real number}} p_0 + (t_2-t_1)p_1 + (2t_2+t_1)p_2$$

polynomial function

$(g(p))(x) := p(x^2+1)$

map from (C).

$B := \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, $C = [p_0+p_1-p_2, p_0+p_1, p_0-p_1]$

$D = \{p_0, p_1, p_2, p_3, p_4\}$

Direct way of computing $[g \circ f]_{D \leftarrow B}$.

$$(g \circ f)\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right)(x) = (t_1 + t_2)\, p_0(x)$$

$$+ (t_2 - t_1)\, p_1(x^2+1)$$

$$+ (2t_2 + t_1)\, p_2(x^2+1)$$

$$= t_1 + t_2 + (t_2 - t_1)(x^2+1)$$

$$+ (2t_2 + t_1)(x^4 + 2x^2 + 1)$$

$$= (4t_2 + t_1) + (5t_2 + t_1)x^2$$

$$+ (2t_2 + t_1)\, x^4$$

$$(g \circ f)\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)(x) = 5 + 6x^2 + 3x^4$$

$$(g \circ f)\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)(x) = -3 - 4x^2 - x^4$$

So $$[g \circ f]_{D \leftarrow B} = \begin{pmatrix} 5 & -3 \\ 0 & 0 \\ 6 & -4 \\ 0 & 0 \\ 3 & -1 \end{pmatrix}$$

Way using compositions of matrices:

$$[g \circ f]_{D \leftarrow B} = [g]_{D \leftarrow C}\ [f]_{C \leftarrow B}$$

We have $[f]_{C \leftarrow B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \\ 1 & 1 \end{pmatrix}$

$$[g]_{D \leftarrow C} = [g]_{D \leftarrow \{P_0, P_1, P_2\}} \circ [id_{P_2}]_{\{P_0, P_1, P_2\} \leftarrow C}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

So $[g \circ f]_{D \leftarrow B} = [g]_{D \leftarrow C} \circ [f]_{C \leftarrow B}$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} -3 & 1 \\ 4 & -2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -3 \\ 0 & 0 \\ 6 & -4 \\ 0 & 0 \\ 3 & -1 \end{pmatrix}$$

<u>Remark 333:</u> (polynomial functions and polynomials)

<u>polynomial function</u>: $p : \mathbb{R} \to \mathbb{R}$, $p(t) = \sum_{i=0}^{n} a_i t^i$

<u>Ex</u>: $p : \mathbb{R} \to \mathbb{R}$, $p(t) = 2t^2 + t - 1$

<u>polynomial</u>: expression of the form
$$a_n \mathbb{X}^n + a_{n-1} \mathbb{X}^{n-1} + \dots + a_1 \mathbb{X} + a_0$$

<u>Ex</u>: $2\mathbb{X}^2 + \mathbb{X} - 1$

A <u>polynomial</u> $Q(\mathbb{X})$ defines a <u>polynomial function</u>
$$\mathbb{R} \longrightarrow \mathbb{R}$$
$$t \longmapsto Q(t)$$

<u>Ex</u>: The polynomial
$$Q(\mathbb{X}) = \mathbb{X}^2 + 2\mathbb{X} - 3$$
defines $q : \mathbb{R} \longrightarrow \mathbb{R}$
$$q(t) := Q(t) = t^2 + 2t - 3$$

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Look at Example 332 (c) in terms of polynomials.

$$P_2 = \mathbb{R} X^0 + \mathbb{R} X + \mathbb{R} X^2 \supseteq \underset{\text{basis}}{\{1, X, X^2\}}$$

$$P_4 = \mathbb{R} X^0 + \mathbb{R} X + \mathbb{R} X^2 + \mathbb{R} X^3 + \mathbb{R} X^4$$
$$\supseteq \{1, X, X^2, X^3, X^4\} \text{ basis}$$

$$g : P_2 \longrightarrow P_4 \quad g(Q(X)) := Q(X^2 + 1)$$

$$\text{e.g. } g(X^2 + 2) = (X^2+1)^2 + 2$$
$$= X^4 + 2X^2 + 3.$$

$$[g]_{\{1, X, X^2\} \to \{1, X, \dots, X^4\}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In Example 332 (d)

$$[g \circ f]_{\{1, X, \dots, X^4\} \leftarrow B} = \begin{pmatrix} 5 & -3 \\ 0 & 0 \\ 6 & -4 \\ 0 & 0 \\ 3 & -1 \end{pmatrix}$$