Reviewing Problems for final

Math 1112 / 2023 Fall

Chapter 5

- Find a 3×3 matrix A that has eigenvalues $1, -1$ and 0, and for which

$$[1,\ -1,\ 1]^T, \quad [1,\ 1,\ 0]^T, \quad [1,\ -1,\ 0]^T$$

are their corresponding eigenvectors.

- Let $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 0 \\ -5 & 5 & 10 \end{bmatrix}$.

(1) Compute the eigenvalues of A, and find a basis of the eigensapces of A.
(2) Let $\mathrm{adj}(A)$ be the adjoint matrix of A. Find the eigenvalues of $3I_3 + \mathrm{adj}(A)$.

- Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.

(1) $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$, (2) $p(\lambda) = \lambda^4 - \lambda^3 + 7$.

- The characteristic polynomial of a matrix A is given. Find the size of the matrix and the possible dimensions of its eigenspaces.

(1) $\lambda^2(\lambda - 1)(\lambda - 2)^3$; (2) $\lambda^3 - 3\lambda^2 + 3\lambda - 1$.

- Compute the characteristic polynomial of

$$A = \begin{bmatrix} & & & 2 \\ 1 & & & 3 \\ & 1 & & -1 \\ & & 1 & -3 \end{bmatrix}.$$

Is it diagonalizable?

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that

(1) A is diagonalizable if $(a-d)^2 + 4bc > 0$;
(2) A is not diagonalizable if $(a-d)^2 + 4bc < 0$;

- Show that A and B are not similar, where A and B are given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

- Let

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}.$$

(a) Find the eigenvalues of A.
(b) For each eigenvalue λ, find the rank of the matrix $\lambda I - A$.
(c) Is A diagonalizable? Justify your conclusion.

- Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

Find A^{-2301} and $\sum_{k=1}^{99} A^k$.

Chapter 6

• Let V be an inner product space. Show that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal unit vectors in V, then $\|\mathbf{u}-\mathbf{v}\| = \sqrt{2}$.

• Do there exists scalers k and l such that the vectors

$$\mathbf{p}_1 = 2 + kx + 6x^2, \quad \mathbf{p}_2 = l + 5x + 3x^2, \quad \mathbf{p}_3 = 1 + 2x + 3x^2$$

are mutually orthogonal with respect to the standard inner product on P_2?

• On P_2, define

$$\langle p(x),\, q(x)\rangle = p(1)q(1) + p'(1)q'(1) + p''(1)q''(1).$$

(1) Prove that $\langle\cdot,\cdot\rangle$ is an inner product on P_2.
(2) Apply the Gram-Schmidt process to transform the standard basis $\{p_1(x), p_2(x), p_3(x)\}$ into an orthonormal basis $\{h_1(x), h_2(x), h_3(x)\}$.

• Find a basis for the orthogonal complement of the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\mathbf{v}_1 = (1,4,5,2), \quad \mathbf{v}_2 = (2,1,3,0), \quad \mathbf{v}_3 = (-1,3,2,2).$$

Here we use Euclidean inner product.

• Let the inner product on P_2 be given by $\langle A, B\rangle = \mathsf{tr}(B^T A)$. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and $W = \mathsf{span}\{A, B\}$.
(1) Verify that A and B are orthogonal.
(2) Find $C_1 \in W$ and $C_2 \in W^\perp$ such that $C = C_1 + C_2$.

• Find the QR-decomposition of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

2. Find the least squares solution of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 9 \\ 3 \end{bmatrix}.$$

And find the least squares error vector and least squares error. Besides, verify that the least squares error is orthogonal to the column space of A.

- Suppose we have observed the following data.

a_i	1	-1	2	3	-2	-3	0
b_i	1.5	1	4.5	8	-2	-4	0.5

Using the least square method, find the affine regression line r (of the form $r(a) = b$) for the above data.

- Find the orthogonal projection of $\mathbf{u} = (2, 1, 3)$ on the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (1, 2, 1)$.

Chapter 7

- Find all values of a, b, c such that the matrix

$$\begin{bmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ c & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

is orthogonal.

- Find a matrix P that orthogonally diagonalizes

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

- Let $Q_A(\mathbf{x}) = 2x_1^2 + x_2^2 + ax_3^2 + 2x_1x_2 + 2bx_1x_3 + 2x_2x_3$, where A is the symmetric matrix associated with $Q_A(\mathbf{x})$. Assume that $\mathbf{v} = (1, 1, 1)$ is an eigenvector of A.

(1) Find the value of a and b.

(2) Find an orthogonal change of variables $\mathbf{y} = P^\top \mathbf{x}$, find the expression $Q_A(\mathbf{y})$.

(3) Determine whether A is positive definite.

- (1) Express the quadratic form $(c_1x_1 + c_2x_2 + \ldots + c_nx_n)^2$ in the matrix notation $x^T A\mathbf{x}$ for A symmetric.

(2) Can the above quadratic form be positive definite?

- (1) Determine the definiteness of the following symmetric matrices.

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} c & a & b \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} (a, b, c \in \mathbb{R}), \quad \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -3 \end{bmatrix}.$$

(2) Show that a positive definite symmetric matrix must have positive determinant.

Chapter 8

- Given $\mathbf{v}_0 = (1, -1, 0) \in \mathbb{R}^3$, we define $T : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$T(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0.$$

(1) Show that T is linear.
(2) Compute the kernel of T, and use the dimension equality to compute the dimension of the range of T.
(3) Compute the range of T directly, and find its dimension.
(4) Compute the matrix for T relative to the standard basis of $\mathbb{R}^3$, and use it to find the rank and nullity of T.

- Let $A = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 1 & 6 & 2 & 1 \\ -3 & 0 & 7 & 1 \end{bmatrix}$ be the matrix for $T : \mathbb{R}^4 \to \mathbb{R}^3$ relative to the bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ and $B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 9 \\ 4 \\ 2 \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 0 \\ 8 \\ 8 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -7 \\ 8 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} -6 \\ 9 \\ 1 \end{bmatrix}.$$

(a) Find $[T(\mathbf{v}_i)]_{B'}$ for $1 \le i \le 4$.
(b) Find $T(\mathbf{v}_i)$ for $1 \le i \le 4$.
(c) Find a formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right)$, and evaluate $T\left(\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}\right)$.

- Let $T_1 : P_1 \to P_2$ be the linear transformation defined by

$$T_1(a_0 + a_1 x) = 2a_0 - 3a_1 x.$$

Let $T_2 : P_2 \to P_3$ be the linear transformation defined by

$$T_2(a_0 + a_1 x + a_2 x^2) = 3a_0 x + 3a_1 x + 3a_2 x^3.$$

Let $B = \{1, x\}$, $B'' = \{1, x, x^2\}$, $B' = \{1, x, x^2, x^3\}$.
(1) Compute $[T_2 \circ T_1]_{B',B}$.
(2) Compute $[T_2]_{B',B''}$ and $[T_1]_{B'',B}$.
(3) What is the relation between $[T_2 \circ T_1]_{B',B}$, $[T_2]_{B',B''}$ and $[T_1]_{B'',B}$.

- Determine whether the matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator T^{-1}, and find $T^{-1}(w_1, w_2, w_3)$.

$$T(x_1, x_2, x_3) = \begin{bmatrix} x_1 - 2x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}.$$

• Suppose that V has basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, and $T : V \to V$ is a linear operator given by $T(\mathbf{v}_i) = \mathbf{v}_{i+1}$ for $i = 1, \ldots, n-1$ and $T(\mathbf{v}_n) = \mathbf{0}$.

(1) Find the matrix representation of T relative to the basis B.

(2) Prove that $T^n = 0$, $T^{n-1} \neq 0$. Here $T^n = T \circ T \circ \ldots \circ T$ for n copies of T.

• (1) Let $f : V \to W$ and $g : W \to U$ be linear maps. Show that $g \circ f : V \to U$ is a linear transformation.

(2) Consider $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) = (2x_1 + 3x_3, 2x_3 - x_2, -x_3 - x_2 - 2x_1, 4x_1 + 3x_2)$. Compute the kernel and range of f.

(3) Consider $g : C[0, 1] \to \mathbb{R}^3$, $g(\varphi) = (\varphi(0), \varphi(0.5), \varphi(1))$. Prove that g is a linear transformation.

(4) Let $W = \{\varphi \in C[0, 1] : \varphi(0) = \varphi(1)\}$. Ler $U = g(W) = \{g(\varphi) : \varphi \in W\}$. Show that $U \cap \mathsf{Ker}(f) = \{0\}$.

SVD

- Find the singular values of A, B and C, where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}.$$

- (1) Prove that the singular values of $A^T A$ are the squares of the singular values of A.

(2) Prove that if $A = U\Sigma V^T$ is a singular value decomposition of A, then U orthogonally diagonalizes $A^T A$.