Reviewing Problems for final

Math 1112 / 2025 Fall

Chapter 5

Notation 1. *With $M_{m\times n}(\mathbb{R})$ we denote the set of real matrices with m rows and n columns.*

Problem 1 (Linear Recursion)**.** *Let us consider the linear recursion equations $x_{k+1} = \frac{x_k+3y_k}{2}$ and $y_{k+1} = \frac{x_k-y_k}{2}$ for $k \in \mathbb{N}$ with $x_1 = y_1 = 1$. Find explicit expressions for x_k and y_k in dependence on k.*

Problem 2 (Eigenfunctions)**.** *Let P_2 denote the set of real-valued polynomials with degree ≤ 2. Let $T : P_2 \to P_2$ be given by*

$$\forall p \in P_2,\ Tp = \frac{d}{dx}\left((x^2+x+1)\frac{dp}{dx}\right).$$

A non-zero function $q \in P_2$ is called a real eigenfunction of T if $Tq = \lambda q$ for a constant $\lambda \in \mathbb{R}$. Find all real eigenfunctions of T.

Problem 3. *Find a 3×3 matrix A that has eigenvalues $1, -1$ and 0, and for which*

$$(1,\ -1,\ 1)^T, \quad (1,\ 1,\ 0)^T, \quad (1,\ -1,\ 0)^T$$

are their corresponding eigenvectors.

Problem 4. *Let* $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 0 \\ -5 & 5 & 10 \end{pmatrix}$.

1. *Compute the eigenvalues of A, and find a basis of the eigensapces of A.*
2. *Let $adj(A)$ be the adjoint matrix of A. Find the eigenvalues of $3I_3 + adj(A)$.*

Problem 5. *Find $\det(A)$ given that A has $p(\lambda)$ as its characteristic polynomial.*

1. $p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$
2. $p(\lambda) = \lambda^4 - \lambda^3 + 7.$

Problem 6. *The characteristic polynomial of a matrix A is given. Find the size of the matrix and the possible dimensions of its eigenspaces.*

1. $\lambda^2(\lambda - 1)(\lambda - 2)^3$;

2. $\lambda^3 - 3\lambda^2 + 3\lambda - 1$.

Problem 7. *Compute the characteristic polynomial of*

$$A = \begin{pmatrix} & & & 2 \\ 1 & & & 3 \\ & 1 & & -1 \\ & & 1 & -3 \end{pmatrix}.$$

Is it diagonalizable?

Problem 8. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$. *Show that*

1. *A is diagonalizable over the real numbers if* $(a-d)^2 + 4bc > 0$;
2. *A is not diagonalizable over the real numbers if* $(a-d)^2 + 4bc < 0$;

Problem 9. *Show that A and B are not similar, where A and B are given by*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Problem 10. *Let*

$$A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}.$$

1. *Find the eigenvalues of A.*
2. *For each eigenvalue* λ, *find the rank of the matrix* $\lambda I - A$.
3. *Is A diagonalizable? Justify your conclusion.*

Problem 11. *Let*

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

Find A^{-2301} *and* $\sum_{k=1}^{99} A^k$.

Problem 12. *Let A and B be two real square matrices of order n and A is similar to B. Then which of the following are/is true?*

(A) *A and B have the same eigenvalues.*

(B) *A and B have the same null spaces.*

(C) *nullity(A) = nullity(B).*

(D) *A and B have the same determinant.*

Problem 13. *Let A be a 3×3 matrix. Suppose that $tr(A) = -5$ and $A^2+2A-3I_3$ is the zero matrix. Find all of the eigenvalues of A.*

Problem 14. *Let $V = M_{2\times 2}(\mathbb{R})$ be the space of all real matrices of order 2. Let $T : V \to V$ be a linear operator on V such that*

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & c \\ b & a \end{pmatrix}.$$

Let $B = \left\{A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\right\}$ *be a basis of V.*

1. *Compute the matrix $[T]_{B\leftarrow B}$ (also denoted by $[T]_B$).*

2. *Is $[T]_B$ diagonalizable? If yes, find a diagonal matrix D and a invertible matrix P such that $D = P^{-1}[T]_BP$; if no, explain the reason.*

3. *Does T have real eigenvalues (Recall that $\lambda \in \mathbb{R}$ is an eigenvalue of T if there is a nonzero vector $\boldsymbol{v} \in V$ such that $T(\boldsymbol{v}) = \lambda\boldsymbol{v}$) ? If yes, find all of them together with a corresponding eigenvector; if no, explain the reason.*

Problem 15. *Find the complex eigenvalues with a corresponding complex eigenvector for the following matrices.* $A = \begin{pmatrix} 4 & -5 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$.

Problem 16. *Find eigenvalues and bases for the eigenspaces of the following matrices*

$$A = \begin{pmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{pmatrix}, \quad C = B^2-2B-3I_3, \quad D = B^{-2}.$$

Problem 17. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *such that $ad-bc = 1$ and $|a+d| > 2$. Investigate whether A is similar to a diagonal matrix.*

Problem 18. *Let* $A = \begin{pmatrix} 1/2 & 2 & 5 \\ 0 & 0 & -1 \\ 0 & 0 & -1/3 \end{pmatrix}$. *Compute A^{2025}.*

Problem 19. *Let* $A = \begin{pmatrix} 3 & 2 & -2 \\ k & 1 & -k \\ 4 & 2 & -3 \end{pmatrix}$, *where k is a real parameter. Find all possible k such that A is diagonalizable.*

Problem 20. *Let* $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 0 \\ -5 & 5 & 10 \end{pmatrix}$.

1. *Compute the eigenvalues of A, and find a basis of the eigensapces of A.*

2. *Let adj(A) be the adjoint matrix of A. Find the eigenvalues of $3I_3 + \text{adj}(A)$.*

Problem 21. *If a matrix $C \in M_{n\times n}(\mathbb{R})$ satisfies $C^2 = C$, then we call it an idempotent matrix. Now assume $A \in M_{n\times n}(\mathbb{R})$ is an idempotent matrix.*

1. *Prove that $I_n - A$ is an idempotent matrix.*
2. *Prove that $\mathsf{rank}(A) + \mathsf{rank}(I_n - A) = n$.*
3. *Prove that A has $\mathsf{rank}(A)$ linearly independent eigenvectors correponding to the eigenvalue 1 and has $\mathsf{rank}(I_n - A)$ linearly independent eigenvectors correponding to the eigenvalue 0.*

Chapter 6

Problem 22. *The orthogonal projection of* $u = (1, -6, 1)$ *on the subspace of* $\mathbb{R}^3$ *spanned by the vectors* $v_1 = (-1, 2, 1)$ *and* $v_2 = (2, 2, 4)$ *is* (____, ____, ____).

Problem 23. *Find the QR decomposition of the matrix*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Problem 24. *Let* V *be an inner product space. Show that if* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal unit vectors in* V*, then* $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$.

Problem 25. *Do there exists real scalars* k *and* l *such that the vectors*

$$\mathbf{p}_1 = 2 + kx + 6x^2, \quad \mathbf{p}_2 = l + 5x + 3x^2, \quad \mathbf{p}_3 = 1 + 2x + 3x^2$$

are mutually orthogonal with respect to the standard inner product on P_2*, i.e. the inner product given by the Gram matrix* I_3 *with respect to the basis* $\{1, x, x^2\}$*?*

Problem 26. *On* P_2*, define*

$$\langle p(x),\, q(x)\rangle = p(1)q(1) + p'(1)q'(1) + p''(1)q''(1).$$

1. *Prove that* $\langle\cdot,\cdot\rangle$ *is an inner product on* P_2.
2. *Apply the Gram-Schmidt process to transform the standard basis* $\{p_1(x), p_2(x), p_3(x)\}$ *into an orthonormal basis* $\{h_1(x), h_2(x), h_3(x)\}$.

Problem 27. *Find a basis for the orthogonal complement of the subspace of* $\mathbb{R}^4$ *spanned by the vectors*

$$\mathbf{v}_1 = (1, 4, 5, 2), \quad \mathbf{v}_2 = (2, 1, 3, 0), \quad \mathbf{v}_3 = (-1, 3, 2, 2).$$

Here we use Euclidean inner product.

Problem 28. *Let the inner product on* $M_{2\times 2}(\mathbb{R})$ *be given by* $\langle A, B\rangle = \mathsf{tr}(B^T A)$. *Let*

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and $W = \mathsf{span}\{A, B\}$.

1. *Verify that* A *and* B *are orthogonal to each other.*
2. *Find* $C_1 \in W$ *and* $C_2 \in W^\perp$ *such that* $C = C_1 + C_2$.

Problem 29. *Find the QR-decomposition of*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Problem 30. *Find the least squares solution of $A\mathbf{x} = \mathbf{b}$, where*

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 0 \\ 9 \\ 3 \end{pmatrix}.$$

And find the least squares error vector and least squares error. Besides, verify that the least squares error is orthogonal to the column space of A.

Problem 31. *Suppose we have observed the following data.*

a_i	1	−1	2	3	−2	−3	0
b_i	1.5	1	4.5	8	−2	−4	0.5

Using the least square method, find the affine regression line r (of the form $r(a) = b$) for the above data.

Problem 32. *Find the orthogonal projection of $\mathbf{u} = (2, 1, 3)$ on the subspace of $\mathbb{R}^3$ spanned by the vectors $\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (1, 2, 1)$.*

Chapter 7

Problem 33. *Find all vectors* $(a, b, c) \in \mathbb{R}^3$ *such that the matrix*

$$\begin{pmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ c & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

is orthogonal.

Problem 34. *Find a matrix* P *that orthogonally diagonalizes*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Problem 35. *Let* $Q_A(\mathbf{x}) = 2x_1^2 + x_2^2 + ax_3^2 + 2x_1x_2 + 2bx_1x_3 + 2x_2x_3$, *where* A *is the symmetric matrix associated with* $Q_A(\mathbf{x})$. *Assume that* $\mathbf{v} = (1, 1, 1)$ *is an eigenvector of* A.

1. *Find the value of* a *and* b.
2. *Find an orthogonal change of variables* $\mathbf{y} = P^\top \mathbf{x}$, *such that* $Q_A(P\mathbf{y})$. *is diagonal, i.e. it has no cross product terms with respect to* $\mathbf{y}$. *Find the expression* $Q_A(P\mathbf{y})$.
3. *Determine whether* A *is positive definite.*

Problem 36. 1. *Express the quadratic form* $(c_1x_1 + c_2x_2 + \ldots + c_nx_n)^2$ *in the matrix notation* $x^T A\mathbf{x}$ *for* A *symmetric.*

2. *Can the above quadratic form be positive definite?*

Problem 37. 1. *Determine the definiteness of the following symmetric matrices.*

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} c & a & b \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} (a, b, c \in \mathbb{R}), \quad \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -3 \end{pmatrix}.$$

2. *Show that a positive definite symmetric matrix must have positive determinant.*

Chapter 8

Problem 38. *Given* $\mathbf{v}_0 = (1, -1, 0) \in \mathbb{R}^3$, *we define* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *as*

$$T(\mathbf{v}) = \mathbf{v} \times \mathbf{v}_0.$$

1. *Show that* T *is linear.*
2. *Compute the kernel of* T, *and use the dimension equality to compute the dimension of the range of* T.
3. *Compute the range of* T *directly, and find its dimension.*
4. *Compute the matrix for* T *relative to the standard basis of* $\mathbb{R}^3$, *and use it to find the rank and nullity of* T.

Problem 39. *Let* $A = \begin{pmatrix} 3 & -2 & 1 & 0 \\ 1 & 6 & 2 & 1 \\ -3 & 0 & 7 & 1 \end{pmatrix}$ *be the matrix for* $T : \mathbb{R}^4 \to \mathbb{R}^3$ *relative to the bases* $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ *and* $B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, *where*

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 6 \\ 9 \\ 4 \\ 2 \end{pmatrix},$$

$$\mathbf{w}_1 = \begin{pmatrix} 0 \\ 8 \\ 8 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} -7 \\ 8 \\ 1 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} -6 \\ 9 \\ 1 \end{pmatrix}.$$

(a) *Find* $[T(\mathbf{v}_i)]_{B'}$ *for* $1 \le i \le 4$.

(b) *Find* $T(\mathbf{v}_i)$ *for* $1 \le i \le 4$.

(c) *Find a formula for* $T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}\right)$, *and evaluate* $T\left(\begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}\right)$.

Problem 40. *Let* $T_1 : P_1 \to P_2$ *be the linear transformation defined by*

$$T_1(a_0 + a_1 x) = 2a_0 - 3a_1 x.$$

Let $T_2 : P_2 \to P_3$ *be the linear transformation defined by*

$$T_2(a_0 + a_1 x + a_2 x^2) = 3a_0 x + 3a_1 x^2 + 3a_2 x^3.$$

Let $B = \{1, x\}$, $B'' = \{1, x, x^2\}$, $B' = \{1, x, x^2, x^3\}$.

1. *Compute* $[T_2 \circ T_1]_{B' \leftarrow B}$.

2. *Compute* $[T_2]_{B'\leftarrow B''}$ *and* $[T_1]_{B''\leftarrow B}$.

3. *What is the relation between* $[T_2 \circ T_1]_{B'\leftarrow B}$, $[T_2]_{B'\leftarrow B''}$ *and* $[T_1]_{B''\leftarrow B}$.

Problem 41. *Determine whether the matrix transformation* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator* T^{-1}, *and find* $T^{-1}(w_1, w_2, w_3)$.

$$T(x_1, x_2, x_3) = \begin{pmatrix} x_1 - 2x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_2 \end{pmatrix}.$$

Problem 42. *Suppose that* V *has basis* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, *and* $T : V \to V$ *is a linear operator given by* $T(\mathbf{v}_i) = \mathbf{v}_{i+1}$ *for* $i = 1, \ldots, n-1$ *and* $T(\mathbf{v}_n) = \mathbf{0}$.

1. *Find the matrix representation of* T *relative to the basis* B.

2. *Prove that* $T^n = 0$, $T^{n-1} \neq 0$. *Here* $T^n = T \circ T \circ \ldots \circ T$ *for* n *copies of* T.

Problem 43.

1. *Let* $f : V \to W$ *and* $g : W \to U$ *be linear maps. Show that* $g \circ f : V \to U$ *is a linear transformation.*

2. *Consider* $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) = (2x_1 + 3x_3, 2x_3 - x_2, -x_3 - x_2 - 2x_1, 4x_1 + 3x_2)$. *Compute the kernel and range of* f.

3. *Consider* $g : C[0,1] \to \mathbb{R}^3$, $g(\varphi) = (\varphi(0), \varphi(0.5), \varphi(1))$. *Prove that* g *is a linear transformation.*

4. *Let* $W = \{\varphi \in C[0,1] : \varphi(0) = \varphi(1)\}$. *Ler* $U = g(W) = \{g(\varphi) : \varphi \in W\}$. *Show that* $U \cap \mathsf{Ker}(f) = \{0\}$.

SVD

Problem 44. *Find the singular values of* A, B *and* C, *where*

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix}.$$

Problem 45.

1. *Prove that the singular values of* $A^T A$ *are the squares of the singular values of* A.

2. *Prove that if* $A = U\Sigma V^T$ *is a singular value decomposition of* A, *then* U *orthogonally diagonalizes* AA^T.