Chapter 7

Spectral decomposition and quadratic forms.

VII 1. Orthogonal matrices

Def 283: $A \in \mathbb{R}^{n\times n}$ is called orthogonal if $A^T A = I_n$.

Prop 284: Let $A \in \mathbb{R}^{n\times n}$. T.a.e.:
1° A is orthogonal
2° The columns form an ON-basis of $\mathbb{R}^{n\times 1}$ wrt. $\circ$-product.
3° The rows form an ON-basis of $\mathbb{R}^{n}$ wrt. $\circ$-product.
4° A is invertible and $A^{-1} = A^T$.
5° A^T is orthogonal.

Proof: Exercise. $\square$

Notation 285: $M_n(\mathbb{R}) := \mathbb{R}^{n\times n}$
$GL_n(\mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\}$
$O_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid A^T A = I_n\}$.

<u>Example 286:</u> a) $I_n \in O_n(\mathbb{R})$

$(I_n^T I_n = I_n)$,

$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \in O_2(\mathbb{R})$

$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}^T \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = I_2$

$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in O_2(\mathbb{R})$, $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = I_2$.

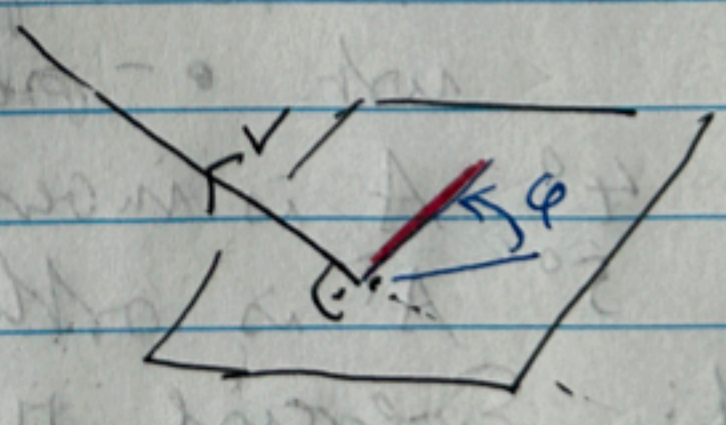
b) $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ satisfies

$R(\varphi)^T R(\varphi) = \begin{pmatrix} \cos^2\varphi + \sin^2\varphi & 0 \\ 0 & 1 \end{pmatrix} = I_2$

$\Rightarrow R(\varphi) \in O_2(\mathbb{R})$.

c) $R_v(\varphi)$ $\quad v \in \mathbb{R}^{3\times 1} \setminus \{\underline{0}\}$

rotation



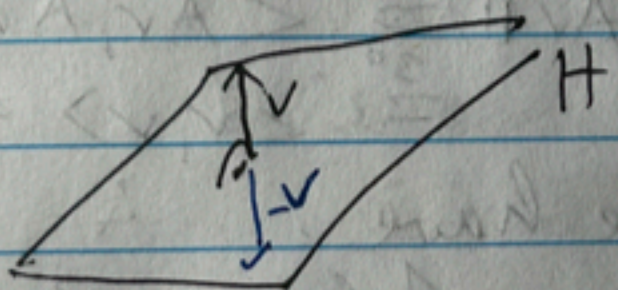
$R_v(\varphi) \in O_3(\mathbb{R})$.

Exercise.

d) $H \leq \mathbb{R}^n$ a hyperplane (through 0)

$0 \neq v \perp H$.
Take $A \in M_n(\mathbb{R})$ s.t.

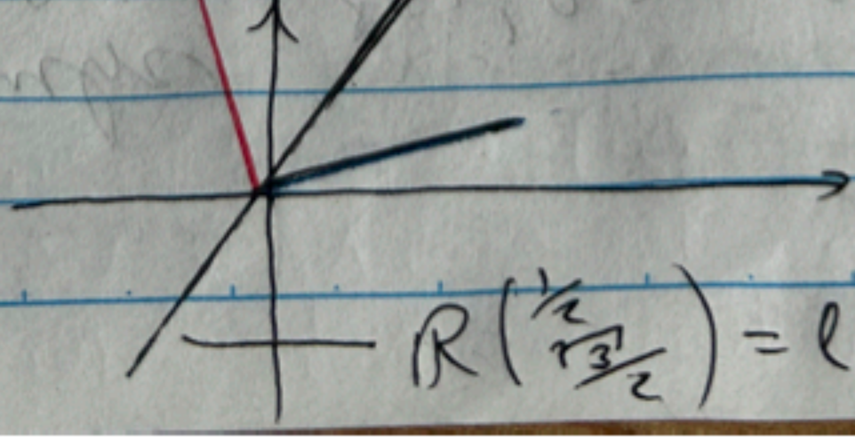
$$Aw = w \ \forall w \in H \text{ and } Av = -v$$



T_A is a reflection about H.
Then $A \in O_n(\mathbb{R})$. (next prop.)

Ex: $A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

or $A = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ reflection about l.



$\mathbb{R}\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = l$

Date Convention 287: In Chapter 7: $\langle , \rangle = \langle , \rangle_{I_n}$.

Prop. 288: Let $A \in M_n(\mathbb{R})$. T.a.e.

1° $A \in O_n(\mathbb{R})$

2° $\|Av\| = \|v\| \ \forall v \in \mathbb{R}^{n \times 1}$

3° $\langle Av, Aw \rangle = \langle v, w \rangle \ \forall v, w \in \mathbb{R}^{n \times 1}$

Proof: $1^\circ \Rightarrow 3^\circ$ $\langle Av, Aw \rangle = v^T A^T A w$
$\overset{1^\circ}{=} v^T I_n w = \langle v, w \rangle$

$3^\circ \Rightarrow 2^\circ$ $\|Av\|^2 = \langle Av, Av \rangle$
$\overset{3^\circ}{=} \langle v, v \rangle = \|v\|^2$

$2^\circ \Rightarrow 3^\circ$. We have
$\langle v, w \rangle = \frac{1}{4}(\|v+w\|^2 - \|v-w\|^2)$

Thus $\langle Av, Aw \rangle = \frac{1}{4}(\|A(v+w)\|^2 - \|A(v-w)\|^2)$
$\overset{2^\circ}{=} \langle v, w \rangle$.

$3^\circ \Rightarrow 1^\circ$ $B := A^T A = (b_{ij})_{ij}$

$b_{ij} = \langle A\vec{e}_i, A\vec{e}_j \rangle$
$\underset{3^\circ}{=} \langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}$

$\left(\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}\right.$ Kronecker symbol. □

VII.2. Orthogonal similarity and spectral decomposition

Def 289: $A, B \in M_n(\mathbb{R})$ are called orthogonally similar if $\exists\, Q \in O_n(\mathbb{R})$
$Q^T A Q = B$.
Write $A \overset{o}{\sim} B$

Remark 290: $A \overset{o}{\sim} B \Rightarrow A \sim B$.
(because, for $Q \in O_n(\mathbb{R})$, $Q^T = Q^{-1}$)

$A \overset{o}{\sim} B$ means we get B from A by orthogonal coordinate transformation.

$$\text{coo}: \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = P \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{\text{new coo.}}$$

(those new coo. are along the columns of P.

$$[T_A]_{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}} = P^{-1} A P \underset{\text{if } P^T = P^{-1}}{=} P^T A P \, .$$

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Theorem 291: Let $A \in M_n(\mathbb{R})$. T.a.e.:

1° $A = A^T$

2° $\exists\ D \in M_n(\mathbb{R})$ diagonal:

$A \overset{\circ}{\sim} D$.

Proof: $2^\circ \Rightarrow 1^\circ$ If $\exists\ Q \in O_n(\mathbb{R})$:

$Q^T A Q = D$ diag.

$\Rightarrow$ $(\cancel{Q^T A Q}$

$A^T = (Q D Q^T)^T = (Q^T)^T D^T Q^T$

$= Q D Q^T = A$.

$1^\circ \Rightarrow 2^\circ$ Thm 238 $\Rightarrow$ $\mathrm{Spec}(A) \subseteq \mathbb{R}$.

Process: 1) Take basis B_i of $\mathrm{Eig}(A, \lambda_i)$

$(\mathrm{Spec}(A) = \{\lambda_1, \dots, \lambda_\ell\})$

2) Then: Apply Gram-Schmidt on each B_i

$\Rightarrow$ get B_i^{GS}

3) Form $B = B_1^{GS} \dot{\cup} \dots \dot{\cup} B_\ell^{GS}$

$= \{u_1, u_2, \dots, u_n\}$

and $Q := (u_1, \dots, u_n)$

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Then • B is a basis of $\mathbb{R}^{n\times 1}$ (Thm 238)

• $B_i \perp B_j$ for $i \neq j$ (Exercise)

so B is ON.

Thus $Q \in O_n(\mathbb{R})$ and

$$AQ = Q\begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_e \end{pmatrix} \begin{matrix} \}\nu_1 \\ \vdots \\ \}\nu_e \end{matrix} \qquad \square$$

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Def 292: Take $A = A^T = Q D Q^T$

$Q \in O_n(\mathbb{R})$, $D \in M_n(\mathbb{R})$ diag.

$Q = (u_1, \dots, u_n)$, $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

So $A = (\lambda_1 u_1, \dots, \lambda_n u_n) \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix}$

$$= \lambda_1 u_1 u_1^T + \dots + \lambda_n u_n u_n^T \qquad (*)$$

"a spectral decomposition of A".

[Explanation for (*):

$(\boxed{c_1}, \boxed{c_2}, \dots, \boxed{c_n}) = (\boxed{c_1}, 0, \dots, 0) + (0, \boxed{c_2}, 0, \dots, 0) + \dots$

$(\boxed{c_1}, 0, \dots, 0) \begin{pmatrix} \boxed{r_1} \\ \vdots \\ \boxed{r_n} \end{pmatrix} = (\boxed{c_1}, 0, \dots, 0) \begin{pmatrix} \boxed{r_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$= \boxed{c_1} \circ \boxed{r_1}$]

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Example 293:

a) $A = \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} = 1 \cdot \vec{e}_1 \vec{e}_1^T + 2 \vec{e}_2 \vec{e}_2^T$

$= 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

b) $A = A^T$ of rk 1.

Then $A = Q \begin{pmatrix} \lambda & & \\ & 0 & \\ & & 0 \end{pmatrix} Q^T, \; Q \in O_n(\mathbb{R})$

$= \lambda u u^T$

$\|u\| = 1$.

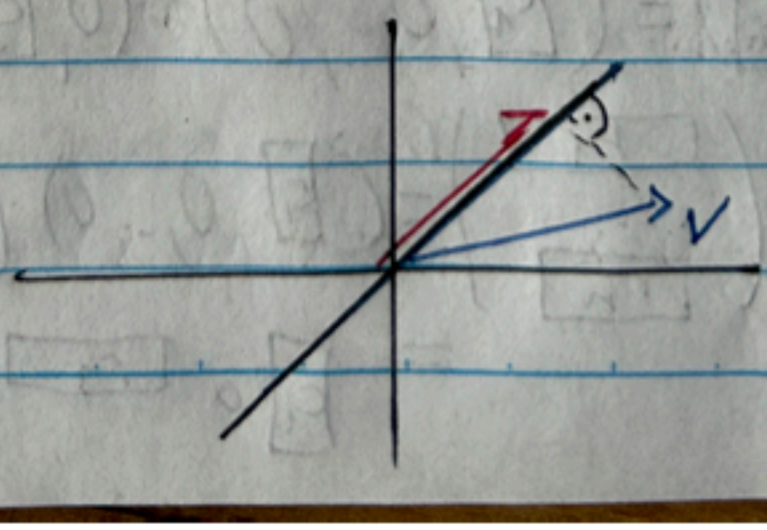
uu^T is the orthogonal projection onto $\mathbb{R}u$:

$(uu^T)u = u(\underbrace{u^T u}_{1}) = u$

If $v \perp u$: $(uu^T)\vec{v} = \underline{0}$.

Ex: $u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad uu^T = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$A := uu^T$. $\quad Av = \begin{pmatrix} \frac{1}{2}(v_1 + v_2) \\ \frac{1}{2}(v_1 + v_2) \end{pmatrix}$



c) So $A = A^T$ projects v onto the "principal axes of A" given by $u_1, \dots, u_n$, scales those projections and adds them.

$$Av = \lambda_1 \, \mathrm{proj}_{u_1}(v) + \dots + \lambda_n \, \mathrm{proj}_{u_n}(v).$$

d) $A = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{pmatrix}$

$$P_A(\lambda) = \lambda^3 + 18\lambda^2 + 81\lambda + 108$$
$$= (\lambda+3)^2(\lambda+12)$$

$$\ulcorner \; P_A(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

$$a_2 = -\mathrm{tr}(A) = -(-4-7-7) = 18$$

$$a_1 = +\left(\begin{vmatrix} -4 & +2 \\ 2 & -7 \end{vmatrix} + \begin{vmatrix} -4 & -2 \\ -2 & -7 \end{vmatrix} + \begin{vmatrix} -7 & 4 \\ 4 & -7 \end{vmatrix} \right)$$
$$= 24 + 24 + 33 = 81$$

$$-a_0 = \det(A) = 2 \begin{vmatrix} -2 & 1 & -1 \\ 0 & -6 & 3 \\ 0 & -3 & -3 \end{vmatrix} = (-4)(-9) \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -4 \cdot 27 = -108 \; \lrcorner$$

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$$\mathrm{Eig}(A, -12) = \mathbb{R}\begin{pmatrix}1\\-2\\2\end{pmatrix} \underset{\text{G-S}}{=} \mathbb{R}\,\frac{1}{\sqrt{9}}\begin{pmatrix}1\\-2\\2\end{pmatrix}$$

$$\mathrm{Eig}(A, -3) = \mathbb{R}\begin{pmatrix}-2\\0\\1\end{pmatrix} + \mathbb{R}\begin{pmatrix}2\\1\\0\end{pmatrix}$$

$$\underset{\text{G-S}}{=} \mathbb{R}\,\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\0\\1\end{pmatrix} + \mathbb{R}\,\frac{1}{3\sqrt{5}}\begin{pmatrix}2\\5\\4\end{pmatrix}$$

$$\left[\begin{pmatrix}2\\1\\0\end{pmatrix} - \frac{1}{5}\left\langle\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right\rangle\begin{pmatrix}-2\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} + \frac{4}{5}\begin{pmatrix}-2\\0\\1\end{pmatrix}\right.$$

$$\left. = \begin{pmatrix}2\\5\\4\end{pmatrix}\frac{1}{5}. \quad \text{Normalize: } \frac{1}{\sqrt{45}}\begin{pmatrix}2\\5\\4\end{pmatrix}\right]$$

$$Q := \begin{array}{ccc} \left(\begin{array}{ccc} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \end{array}\right) \\ \begin{array}{ccc} u_1 & \quad u_2 & \quad u_3 \end{array} \end{array}$$

$$A = -12\, u_1 u^T - 3\, u_2 u_2^T - 3\, u_3 u_3^T$$

$$\begin{pmatrix}1\\-1\\2\end{pmatrix}(1\ -1\ 2)$$

$$= -12 \cdot \frac{1}{9}\begin{pmatrix}1 & -1 & 2\\-1 & 1 & -2\\2 & -2 & 4\end{pmatrix}$$

$$-3 \cdot \frac{1}{5}\begin{pmatrix}4 & 0 & -2\\0 & 0 & 0\\-2 & 0 & 1\end{pmatrix}$$

$$-3 \cdot \frac{1}{45}\begin{pmatrix}4 & 10 & 8\\10 & 25 & 20\\8 & 20 & 16\end{pmatrix}$$

VII 3. Singular value decomposition

Def 294: Let $A \in \mathbb{R}^{m\times n}$ and $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of A^TA. The numbers $\sigma_1 = \sqrt{\lambda_1}, \dots, \sigma_n = \sqrt{\lambda_n}$ are called singular values of A.

Example 295: a) Find the singular values of $\begin{pmatrix}1 & 1\\0 & 1\\1 & 0\end{pmatrix} =: A$.

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$$A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$p_{A^TA}(\lambda) = \lambda^2 - 4\lambda + 3$$

$$= (\lambda - 3)(\lambda - 1)$$

The singular values are

$\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{1} = 1$.

b) If A is symmetric

$A = QDQ^T$ $Q \in O_n(\mathbb{R})$

D diag.

$A^TA = QDQ^TQDQ = QD^2Q^T$

$\Rightarrow \mathrm{Spec}(A^TA) = \{\lambda^2 \mid \lambda \in \mathrm{Spec}(A)\}$

The singular values of A are $|\lambda|$, $\lambda \in \mathrm{Spec}(A)$.

If A is positive semidefinite then the singular values are the eigenvalues of A. (with multiplicities)

Singular values are used to generalize the spectral decomposition

<u>Problem:</u> Find U, V orthogonal such that

$$A = U \Sigma V^T$$

with $\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_k & \\ & 0 & & 0 \end{pmatrix} \in \mathbb{R}^{m \times n}$

$$U \in O_m(\mathbb{R}), V \in O_n(\mathbb{R}).$$

$$U = (u_1, \dots, u_m), \quad V = (v_1, \dots, v_n)$$

$$\Rightarrow A = \sigma_1 u_1 v_1^T + \dots + \sigma_k u_k v_k^T$$

Def 296:

We call above $U \Sigma V = A$ a <u>singular value decomposition of A</u>

and above
$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$
a <u>singular value expansion of A.</u>

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Example 297: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$

(Idea: If we have
$$A = U \Sigma V^T$$
then $A^T A = V \Sigma^T \Sigma V^T$
$$\uparrow$$
$$U^T U = I_m$$

and $AV = U\Sigma$

So 1) Find V orthogonally diag. $A^T A$.

and

2) Take U s.t.
$$AV = U\Sigma \quad)$$

Step 1: Find $V \in O_2(\mathbb{R})$:
$$A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\mathrm{Eig}(A^T A, 1) = \mathbb{R} \overbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}}}^{v_1}$$

$$\mathrm{Eig}(A^T A, 3) = \mathbb{R} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}}_{v_2}$$

Put $V = (v_1, v_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \in O_2(\mathbb{R})$

Step 2: Compute $u_1, \dots, u_n$

Here: u_1, u_2.

$$u_1 := \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{1}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Step 3: Extend to ON basis of $\mathbb{R}^{3\times 1}$.

$u_3 \perp \{u_1, u_2\}$ $\quad \|u_3\| = 1$

$$u_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

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Step 4: Put everything together

$$U = (u_1, u_2, u_3)$$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We get $A = U \Sigma V^T$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \circ \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \circ \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Singular value expansion

$$A = \sqrt{3}\, u_1 v_1^T + 1 \cdot u_2 v_2^T$$

$$= \sqrt{3} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \circ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$+ 1 \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \circ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$$

$$= \sqrt{3} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Interpretation of uv^T, $\|u\| = \|v\| = 1$.

uv^T sends $\mathbb{R}v$ to $\mathbb{R}u$ and $(\mathbb{R}v)^\perp$ to zero.

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Theorem 298: (SVD)

Let $A \in \mathbb{R}^{m\times n}$. Then $\exists\, U \in O_m(\mathbb{R})$ $\exists\, V \in O_n(\mathbb{R})$ such that

$$A = U \Sigma V^T \text{ with}$$

$$\Sigma = \begin{pmatrix} \sigma_1 \ddots \sigma_k & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m\times n}$$

Note that if $\operatorname{rank}(A) = k < n$ then $\sigma_{k+1} = \sigma_{k+2} = \dots = \sigma_n = 0$ and $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k > 0$.
(because $rk(A) = rk(A^T A)$)

Proof: 1) Take $V \in O_n(\mathbb{R})$ such that $V\Sigma^T\Sigma V^T = A^T A$ (Thm 291)
$V = (v_1, \dots, v_n)$

2) Say $rk(A) = k$. Put $u_i = \frac{1}{\sigma_i} A v_i$ for $i = 1, \dots, k$

3) Extend to ON-basis of $\mathbb{R}^{m\times 1}$

$$\{u_1, u_2, \dots, u_k, \overbrace{u_{k+1}, \dots, u_m}^{\text{new}}\}$$

Put $U = (u_1, u_2, \dots, u_m)$.

4) We obtain

$$AV = (Av_1, Av_2, \dots, Av_n)$$

$$= (\sigma_1 u_1, \sigma_2 u_2, \dots, \sigma_k u_k, 0, \dots, 0)$$

(because $Av_j = 0$ for $j > k$ as $\text{null}(A) = \text{null}(A^T A)$)

$$= (\sigma_1 u_1, \sigma_2 u_2, \dots, \sigma_k u_k, 0 u_{k+1}, \dots, 0 u_m)$$

$$= (\sigma_1 u_1, \sigma_2 u_2, \dots, \sigma_k u_k, \sigma_{k+1} u_{k+1}, \dots, \sigma_n u_n)$$

$$= U\Sigma \qquad \square$$

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<u>Example</u>: a) $A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

<u>Step 1</u> (V): $A^T A = (1\ 1\ 0)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = (2)$

$= (1)(2)(1)^T$

$V = (1) = I_1 \quad \sigma_1 = \sqrt{2}$

<u>Step 2</u> ($u_1, \dots, u_k$): Here $k = 1$.

$$u_1 := \frac{1}{\sqrt{2}} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \circ (1)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

<u>Step 3</u> Extend to ON-basis $\{u_1, \dots, u_3\}$:

$$u_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

<u>Step 4</u>: Put everything together

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = (1)$$

$$\Sigma = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} (1)^T = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

VII 4. Quadratic forms

<u>Def 299:</u> a) A linear map $f: \mathbb{R}^n \to \mathbb{R}$ is called <u>linear form</u>. It has the form $f(\underline{x}) = a_1 x_1 + \dots + a_n x_n$ with $a_i := f(\vec{e}_i)$.

b) A <u>quadratic form</u> is a map $q: \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form

$$q(\underline{x}) = a_{11} x_1^2 + a_{22} x_2^2 + \dots + a_{nn} x_n^2 + \sum_{i<j} 2 a_{ij} x_i x_j.$$

with constants $a_{ij} \in \mathbb{R}$.

The matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ $(a_{ij} := a_{ji}$ for $i > j)$ is called the matrix associated to q.

We have

$$q(\underline{x}) = x^T A x =: q_A(\underline{x})$$

with $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ the column vector of the variables.

The terms $2 a_{ij} x_i x_j$ are called <u>cross product terms</u>.

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Date c) A quadratic form with no cross prod terms is called diagonal form.

Rk 300: a) A is determined by q.

$$q(\vec{e}_i) = a_{ii}$$

$$q(\vec{e}_i + \vec{e}_j) = a_{ii} + a_{jj} + 2a_{ij}$$

for $i<j$.

i.e. if $q_A = q_B$ for $A, B \in M_n(\mathbb{R})$ symmetric then $A = B$.

Def 301: We call a quadratic form $q: \mathbb{R}^n \to \mathbb{R}$ positive definite / negative definite / positive semidefinite / negative semidefinite / indefinite if A is positive definite / ... .

Recall that means for

		Spec(A)
pos def	$q(x) > 0 \ \forall x \in \mathbb{R}^n \setminus \{\underline{0}\}$	all $\lambda_i > 0$
pos semidef	$q(x) \geq 0 \ \forall x \in \mathbb{R}^n$	all $\lambda_i \geq 0$
neg def	$q(x) < 0 \ \forall x \in \mathbb{R}^n - \{\underline{0}\}$	all $\lambda_i < 0$
neg semidef	$q(x) \leq 0 \ \forall x \in \mathbb{R}^n$	all $\lambda_i \leq 0$
indefinite	$\exists x, y \in \mathbb{R}^n$: $q(x) > 0 > q(y)$.	$\exists \lambda_{i_1}, \lambda_{i_2}$: $\lambda_{i_1} < 0 < \lambda_{i_2}$

Example 302: a) $q(x_1,x_2) = x_1^2 + x_2^2$

$$= (x_1, x_2) \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{A = I_2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

q is positive definite, because all eigenvalues of A are positive.

b) $q(x_1,x_2) = x_1^2 - x_2^2 = (x_1,x_2)\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

q is indefinite, because $\operatorname{spec}(A) = \{1,-1\}$.

c) $q(x) = 2x^2 \qquad q: \mathbb{R} \to \mathbb{R}$.

$A = (2) \in \mathbb{R}^{1\times 1}$. q positive definite.

d) $q(x_1,x_2,x_3) = x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_3 - x_1x_2 + x_2x_3$

$$= (x_1,x_2,x_3) \underbrace{\begin{pmatrix} 1 & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -2 & \frac{1}{2} \\ 1 & \frac{1}{2} & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$q(1,0,0) = 1 > 0$

$q(0,1,0) = -2 < 0$

So q is indefinite.

e) $q_b(x) = x_1^2 + b x_2^2 + 4 x_1 x_2$.

$q_b : \mathbb{R}^2 \to \mathbb{R}$, $b \in \mathbb{R}$

$$A_b = \begin{pmatrix} 1 & 2 \\ 2 & b \end{pmatrix}$$

$q_b(\vec{e}_1) = 1 > 0$, so q_b cannot be neg definite or neg semidefinite, i.e. A_b has at least one positive eigenvalue

Claim:

b	definiteness
$b<4$	indefinite
$b=4$	pos semidefinite
$b>4$	pos definite

Pf: $A_b = Q_b \begin{pmatrix} \lambda_1^{(b)} & \\ & \lambda_2^{(b)} \end{pmatrix} Q_b^T$, $Q_b \in O_2(\mathbb{R})$

with $\lambda_1^{(b)} > 0$.

Thus: A_b is positive def. $\iff \lambda_2^{(b)} > 0$

$\iff \det A_b \underset{\uparrow}{=} \lambda_1^{(b)} \lambda_2^{(b)} > 0$.

$\det Q^T = \det Q = \pm 1$

- A_b pos semidef $\iff \det A_b \geq 0$
- A_b indefinite $\iff \det A_b < 0$.

$\det(A_b) = b - 4$ $\square$

Directly: $q_b(x_1,x_2) = (x_1+2x_2)^2 + (b-4)x_2^2$

We have for $b>4$: $q(x)>0 \ \forall x \neq \underline{0}$

$b=4$: $q(x) \geq 0 \ \forall x \in \mathbb{R}^2$

$b<4$: $q_b(-2,1) = b-4 < 0 < q_b(1,0)$.

f) $f(x_1,x_2) = e^{x_1} \cdot \dfrac{1}{1-x_2}$ $\quad (x_1,x_2) \in \mathbb{R} \times]-1,1[$.

$$= \left(\sum_{n=0}^{\infty} \frac{x_1^n}{n!} \right) \cdot \left(\sum_{m=0}^{\infty} x_2^m \right)$$

$$= \sum_{l=0}^{\infty} \left(\frac{x_1^0}{0!} \cdot x_2^l + \frac{x_1^1}{1!} x_2^{l-1} + \dots + \frac{x_1^l}{l!} x_2^0 \right)$$

$$= 1 + \overbrace{(x_2 + x_1)}^{\text{linear form}} + \overbrace{\left(x_2^2 + x_1 x_2 + \frac{x_1^2}{2}\right)}^{\text{quadratic form.}}$$

+ higher order terms.

The linear form and the quadratic form help to study f near $(0,0)$.

3 Questions: Q1: What kind of curve is given by "$q(x) = c$"?

Q2: How to determine that q is positive definite?

Q3: $\min_{\|x\|=1} q(x) = ?$

$\max_{\|x\|=1} q(x) = ?$

To study those questions we need to change variables, i.e. an equation of the form

$$x = Py \quad , P \in GL_n(\mathbb{R})$$

with old variables x and new var. y.
If $P \in O_n(\mathbb{R})$, we call it an orthogonal change of variables.

Def 30.3: Two quadratic forms q_1, q_2 are called (orthogonally) equivalent if there is a change of variables (an orthogonal — " —) $x = Py$

which transforms q_1 to q_2, i.e. the map

$$\mathbb{R}^n \ni (t_1, \dots, t_n) \longmapsto q_1\left(P\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}\right)$$

is equal to q_2.
Write $q_1 \sim q_2$ ($q_1 \overset{\sigma}{\sim} q_2$)

__Example 3.04:__ a) $q_1 = q_{A_1}$, $q_2 = q_{A_2}$
$A_1, A_2 \in M_n(\mathbb{R})$ symmetric.
Then (1) $q_1 \sim q_2 \iff \exists P \in GL_n(\mathbb{R})$:
$P^T A_1 P = A_2$.
(2) $q_1 \overset{\sigma}{\sim} q_2 \iff \exists P \in O_n(\mathbb{R})$:
$P^T A_1 P = A_2$.

__Proof:__ (1) "$\Rightarrow$" $q_1 \sim q_2 \iff \exists P \in GL_n(\mathbb{R})$:

$$q_1\left(P\begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}\right) = q_2(t_1, \dots, t_n) \quad \forall t_1, \dots, t_n \in \mathbb{R}$$

$\iff \exists P \in GL_n(\mathbb{R})$: $(t_1, \dots, t_n) P^T A_1 P \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = (t_1, \dots, t_n) A_2 \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$
for all $t_1, \dots, t_n \in \mathbb{R}$

$\underset{\text{Rk } 3.00(a)}{\iff} \exists P \in GL_n(\mathbb{R})$: $P^T A_1 P = A_2$ $\square$

(2) similar

(b) $\overset{q_1}{\overset{\shortparallel}{q_{\begin{pmatrix}1&2\\2&1\end{pmatrix}}}} \sim \overset{q_2}{\overset{\shortparallel}{q_{\begin{pmatrix}6&3\\3&1\end{pmatrix}}}}$

because $\begin{pmatrix}1&1\\&1\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}1&\\1&1\end{pmatrix}=\begin{pmatrix}3&3\\2&1\end{pmatrix}\begin{pmatrix}1&\\1&1\end{pmatrix}$

$$= \begin{pmatrix}6&3\\3&1\end{pmatrix}.$$

Another way:

(EC) $$\left\{\begin{aligned} & x_1^2+4x_1x_2+x_2^2 \overset{!}{=} (x_1+2x_2)^2-3x_2^2 \\ & \underset{\uparrow}{=} \quad y_1^2-y_2^2 \\ & y_1 := x_1+2x_2 \\ & y_2 := \sqrt{3}\,x_2 \end{aligned}\right.$$

(The substitution is invertible, because we get x_1, x_2 back from y_1, y_2.)

and $6x_1^2+6x_1x_2+x_2^2$

$$= (x_2+3x_1)^2-3x_1^2$$

$$\underset{\uparrow}{=} \quad y_1^2-y_2^2.$$

$y_1 := x_2+3x_1$

$y_2 := \sqrt{3}\,x_1$

Thus $x_1^2 + 4x_1x_2 + x_2^2 \sim 6x_1^2 + 6x_1x_2 + x_2^2$.

$$P_1^2 \quad \cancel{x_1}^2 - \cancel{x_2}^2 \quad P_2$$

$$P_1 = \begin{pmatrix} 1 & 2 \\ & \sqrt{3} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\frac{2}{\sqrt{3}} \\ & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 3 & 1 \\ \sqrt{3} & \end{pmatrix}^{-1} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 \\ -\sqrt{3} & 3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ 1 & -\sqrt{3} \end{pmatrix}$$

<u>Question:</u> Are q_1 and q_2 orthogonally equivalent?

Answer: No, because $\operatorname{tr}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 2$
$$\neq$$
$$\operatorname{tr}\begin{pmatrix} 6 & 3 \\ 3 & 1 \end{pmatrix} = 7$$

<u>Question:</u> For which diagonal form is q_1 orthogonally equivalent?

Orthogonally diagonalize $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$:

$$p_A(\lambda) = \lambda^2 - 2\lambda - 3$$

$$\lambda_{1,2} = 1 \pm \sqrt{1+3} = \begin{cases} -1 = \lambda_1 \\ 3 = \lambda_2 \end{cases}$$

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$$(\text{OEC})\left\{\begin{array}{l} \operatorname{Eig}(A,-1) = \mathbb{R}\begin{pmatrix}1\\-1\end{pmatrix} \\ \operatorname{Eig}(A,3) = \mathbb{R}\begin{pmatrix}1\\1\end{pmatrix} \\ P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ P^T A P = \begin{pmatrix} 3 & \\ & -1 \end{pmatrix} \\ \Rightarrow \quad x_1^2 + 4x_1x_2 + x_2^2 \underset{\overrightarrow{P}}{\overset{o}{\sim}} 3y_1^2 - y_2^2 \end{array}\right.$$

We have seen:

(EC) eleminating cross product terms

$$x_1^2 + 4x_1x_2 + x_2^2 \sim y_1^2 - y_2^2$$

(OEC) orthogonally eliminating cross product terms.

($\Leftrightarrow$ orthogonally diagonalizing A)

(c) orthogonally eliminate cross terms of $x_1^2+x_2^2+x_3^2-2x_1x_2-2x_1x_3+2x_2x_3$.

(Exercise!)

<u>Sol^n:</u> $A := \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

$$P_A(\lambda) = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$$

Thus $q \underset{P}{\overset{O}{\sim}} 3x_1^2 + 0x_2^2 + 0x_3^2$

You find a $P \in O_n(\mathbb{R})$ ~~such that~~ by the (OEC) method. □

<u>Theorem 305:</u> (Principal axes theorem)

Let $q \overset{q_A}{=}$ be a quadratic form on $\mathbb{R}^n$. Then there is a diagonal form d which is orthogonally equivalent to q. The coefficients of d are the eigenvalues of A counting multiplicities; i.e. $\lambda \in \mathrm{Spec}(A)$ occurs $m_g(A, \lambda)$ times.

<u>Pf:</u> Theorem 291. □

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This answers Question 3.

$q = q_A \sim d \quad d(x_1, \dots, x_n) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ eigenvalue of A (counting multiplicities)

Then

$$\max_{\|x\|=1} q_A(x) = \lambda_1$$

$$\min_{\|x\|=1} q_A(x) = \lambda_n.$$

Prove: Take ON basis $\{v_1, v_2, \dots, v_n\}$ s.t. $v_i \in Eig(A, \lambda_i)$

Take $v = \sum \mu_i v_i$ s.t. $\|v\| \underset{\uparrow \text{ON}}{=} \sum \mu_i^2 = 1.$

$$d(v) = \sum \mu_i^2 \lambda_i \leq \sum \mu_i^2 \lambda_1 = \lambda_1 = d(v_1)$$

$$d(v) \geq \sum \mu_i^2 \lambda_n = \lambda_n = d(v_n).$$

$\square$

We now approach Question 2.

Def.: Let $A \in M_n(\mathbb{R})$ be symmetric. The submatrices

$$A_{(i)} := \begin{pmatrix} a_{11} & \cdots & a_{1i} \\ \vdots & & \vdots \\ a_{i1} & & a_{ii} \end{pmatrix}$$

are called leading principal submatrices and $M_{(i)} := \det(A_{(i)})$ the leading principal minors.

[Principal submatrix: $J \subseteq \{1, \dots, n\}$

$$A_J := (a_{ij})_{i,j \in J}$$]

Theorem (Sylvester) 3.06: Let $A \in M_n(\mathbb{R})$ be symmetric and $\det(A) \neq 0$.

Then: 1) A positive definite
$\iff M_{(1)}, M_{(2)}, \dots, M_{(n)}$ are all positive

2) A negative definite
$\iff M_{(1)} < 0, M_{(2)} > 0 \wedge M_{(3)} < 0 \wedge M_{(4)} > 0 \wedge \dots \wedge M_{(n)} = \begin{cases} > 0, & 2\mid n \\ < 0, & 2 \nmid n. \end{cases}$

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$\det(A) \neq 0$

3) A is indefinite $\Longleftrightarrow$
A is not positive definite
and A is not negative definite

Example 307:

(a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad M_{(1)} = 1 > 0$

$M_{(2)} = -3 < 0$

$\Rightarrow$ A is indefinite

(b) $A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ $\quad M_{(1)} = 1 > 0$

$$M_{(2)} = 1 - a^2 = \begin{cases} > 0, & |a| < 1 \\ < 0, & |a| > 1 \\ 0, & |a| = 1 \end{cases}$$

So A is positive definite for $|a| < 1$

~~negativ~~

indefinite for $|a| > 1$

For $|a| = 1$ Thm 306 doesn't give an answer.

$|a|=1$: $\quad p_A(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda-2)$

All eigenvalues ≥ 0

$\Rightarrow$ A is positive ~~definite~~ semi definite.

indefinite $\quad -1 \quad$ pos def. $\quad 1 \quad$ ~~pos semidef~~ indefinite

pos semi def.

(c) $\quad A = \begin{pmatrix} 1 & a & \\ a & 1 & 1 \\ & 1 & 1 \end{pmatrix}$

$M_{(1)} = 1 > 0$

$M_{(2)} = 1 - a^2$

$M_{(3)} = \det(A) = \begin{vmatrix} 1 & a & \\ a & 0 & 1 \\ & & 1 \end{vmatrix} = -a^2 \leq 0$

So A is indefinite for $a \neq 0$.
For $a = 0$ Sylvester's criterion does not give an answer.

$a=0$: $\quad p_A(\lambda) = p_{\begin{pmatrix} 1 & & \\ & 1 & 1 \\ & 1 & 1 \end{pmatrix}}(\lambda)$

$= p_{(1)}(\lambda)\, p_{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}(\lambda) = (\lambda-1)(\lambda^2-2\lambda) = \lambda(\lambda-1)(\lambda-2)$, A pos. semidef.

(d) <u>Exercise</u>: Study definiteness of $\begin{pmatrix} 1 & a & \\ a & 1 & 1 \\ & 1 & 2 \end{pmatrix}$ ~~xxx~~.

$M_{(1)} = 1,\quad M_{(2)} = [1 - a^2],\quad M_{(3)} = 1 - 2a^2$

$|a| < \sqrt{\frac{1}{2}}$: A pos. definite
$|a| > \sqrt{\frac{1}{2}}$: A indefinite
$|a| = \sqrt{\frac{1}{2}}$: A pos semidefinite

<u>Proof</u>: Take for $a = \sqrt{\frac{1}{2}}$
$a_n := \sqrt{\frac{1}{2}} - \frac{1}{n}$. Get A_{a_n}

Then for $v \in \mathbb{R}^{n \times 1}$:

$$0 \leq v^T A_{a_n} v \xrightarrow[n \to \infty]{} v^T A_{\sqrt{\frac{1}{2}}} v$$

So $v^T A_{\sqrt{\frac{1}{2}}} v \geq 0$ $\square$.