VII 5. Applications of quadratic forms.

- determine extrema
- conics

VII 5.1 Determine extrema

Def. 308: Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
and $\underline{x}_0 \in \mathbb{R}_n$.
We call $\underline{x}_0$

(a) a local maximum of f if
$\exists \delta > 0$ $\forall$ $\underline{x}$ with $\|\underline{x} - \underline{x}_0\| < \delta$ we have
$f(\underline{x}) \leq f(\underline{x}_0)$

(b) a local minimum of f if
$\underline{x}_0$ is a local maximum of $-f$.

(c) a global maximum (or just called maximum) of f
if $\forall \underline{x} \in \mathbb{R}^n$: $f(\underline{x}) \leq f(\underline{x}_0)$

(d) a global minimum (or just called minimum) of f if
$\forall \underline{x} \in \mathbb{R}^n$: $f(\underline{x}) \geq f(\underline{x}_0)$.

A <u>local extremum</u> of f is a point $\underline{x}_0$ which is a local maximum or a local minimum.

A <u>global extremum</u> of f is a point $\underline{x}_0$ which is a global maximum or a global minimum of f.

<u>Def 309:</u> Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$, $\underline{x}_0 \in \mathbb{R}^n$.
$\underline{x}_0$ is called a <u>stationary point</u> of f if $\frac{\partial f}{\partial x_i}(\underline{x}_0) = 0$ for $i = 1, \dots, n$.

Suppose $f \in C^2(\mathbb{R}^n, \mathbb{R})$. The matrix

$$H_f(\underline{x}) := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

is called the <u>Hessian</u> (or <u>Hessian matrix</u>) of f at $\underline{x}$.

Theorem 3.10: (Second derivative test)

(i) Let $f \in C^1(\mathbb{R}^n, \mathbb{R})$ and suppose f has a local extremum at $\underline{x}_0$. Then $\underline{x}_0$ is a stationary point of f.

(ii) Let $f \in C^2(\mathbb{R}^n; \mathbb{R})$ and $\underline{x}_0$ be a stationary point of f. Then
(a) $\underline{x}_0$ is a local minimum of f if $H_f(\underline{x}_0)$ is positive definite
(b) $\underline{x}_0$ is a local maximum of f if $H_f(\underline{x}_0)$ is negative definite.
(c) $\underline{x}_0$ is not a local extremum if $H_f(\underline{x}_0)$ is indefinite.

Proof: (i) Suppose $\underline{x}_0$ is a local minimum of f (if not, then consider $-f$.)

For $t \in \mathbb{R} \setminus \{0\}$ with $|t|$ small we have

for $t > 0$: $\quad 0 \leq \dfrac{f(\underline{x}_0 + t\vec{e}_i) - f(\underline{x}_0)}{t}$

for $t < 0$: $\quad 0 \geq \dfrac{f(\underline{x}_0 + t\vec{e}_i) - f(\underline{x}_0)}{t}$

So $0 \leq \lim_{t \downarrow 0} \frac{f(\underline{x}_0 + t\vec{e}_i) - f(\underline{x}_0)}{t} = \frac{\partial f}{\partial x_i}(\underline{x}_0)$

and $0 \geq \lim_{t \uparrow 0} \frac{f(\underline{x}_0 + t\vec{e}_i) - f(\underline{x}_0)}{t} = \frac{\partial f}{\partial x_i}(\underline{x}_0)$

Thus $\frac{\partial f}{\partial x_i}(\underline{x}_0) = 0$.

(ii) We just prove (a). ((b) follows from (a) by replacing f by $-f$.)

f is 2-times continuous differentiable. So for $\|h\|$ small we obtain

$$f(\underline{x}_0 + h) = f(\underline{x}_0) + \left(\frac{\partial f}{\partial x_1}(\underline{x}_0), \dots, \frac{\partial f}{\partial x_n}(\underline{x}_0)\right) h$$

$$+ \frac{1}{2} h^T H_f(\underline{x}_0) h + I(h)$$

with $\frac{|I(h)|}{\|h\|^2} \longrightarrow 0$ for $\|h\| \to 0$

$\underline{x}_0$ is a stationary point

$$\Rightarrow f(\underline{x}_0 + h) = f(\underline{x}_0) + \frac{1}{2} h^T H_f(\underline{x}_0) h + I(h)$$

Let λ be the smallest eigenvalue of $H_f(\underline{x}_0)$. Then $\lambda > 0$ (because $H_f(\underline{x}_0)$ is positive definite) and

$$h^T H_f(\underline{x}_0) h \geq \lambda h^T h = \lambda \|h\|^2.$$

(after Thm 305, answer to Q 3)

Consider $0 < \|h\|$ small such that

$$\frac{|\tau(h)|}{\|h\|^2} < \frac{\lambda}{2}$$

Then
$$f(\underline{x}_0 + h) \geq f(\underline{x}_0) + \frac{\lambda}{2}\|h\|^2 - |\tau(h)|$$
$$> f(\underline{x}_0) \text{ for } \|h\| > 0 \text{ small.}$$

$\square$

In fact in (ii) (a) we proved that $\underline{x}_0$ is a "strict" local minimum, i.e. we have

$$f(\underline{x}) > f(\underline{x}_0) \text{ for } \underline{x} \neq \underline{x}_0$$

locally around $\underline{x}_0$.

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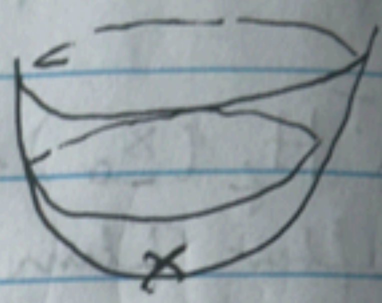
Example 311:

(a) $f(x,y) = x^2 + 2x + y^2 + 1$, $(x,y) \in \mathbb{R}^2$



$\frac{\partial f}{\partial x}(x,y) = 2x+2$

$\frac{\partial f}{\partial y}(x,y) = 2y$

Stationary points: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$\Leftrightarrow \quad (x,y) = (-1,0)$

classification: $H_f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

positive definite

$\Rightarrow$ $(-1,0)$ is a local minimum.

In fact $f(x,y) = \|(x+1,y)\|^2 > \underset{f(-1,0)}{\overset{0}{\shortparallel}}$

for $(x,y) \neq (-1,0)$. So $(-1,0)$ is a global minimum.

(b) $f(x,y) = x^2 + 2y^2 + x^3 y$

$$\frac{\partial f}{\partial x}(x,y) = 2x + 3x^2 y$$

$$\frac{\partial f}{\partial y}(x,y) = 4y + x^3$$

Stationary points: $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$$\begin{cases} 2x + 3x^2 y = 0 \\ 4y + x^3 = 0 \end{cases}$$

Case $x = 0$: $(0,0)$

Case $x \neq 0$: $y = -\frac{1}{4} x^3$ and

$$2 - \frac{3}{4} x^4 = 0$$

$$\Leftrightarrow (x,y) = \pm\left(\sqrt[4]{\frac{8}{3}} , -\frac{1}{4}\left(\frac{8}{3}\right)^{\frac{3}{4}} \right)$$

We have 3 stationary points.

classification: $H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^3 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y \partial y} \end{pmatrix}$

$$= \begin{pmatrix} 2+6xy & 3x^2 \\ 3x^2 & 4 \end{pmatrix}$$

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$(0,0)$: $H_f(0,0) = \begin{pmatrix} 2 & \\ & 4 \end{pmatrix}$

positive definite $\Rightarrow$ $(0,0)$ is a local minimum.

$(x,y) \in \left\{ \left(\sqrt[4]{\frac{8}{3}}, -\frac{1}{4}\left(\frac{8}{3}\right)^{\frac{3}{4}} \right), \left(-\left(\frac{8}{3}\right)^{\frac{1}{4}}, \frac{1}{4}\left(\frac{8}{3}\right)^{\frac{3}{4}} \right) \right.$

Then $H_f(x,y) = \begin{pmatrix} -2 & 2\sqrt{6} \\ 2\sqrt{6} & 4 \end{pmatrix}$

is indefinite (Look at the diagonal entries.)

So (x,y) is not a local extremum (In fact it is a saddle point.)

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(c) $f(x,y) = x^3 - 3xy - y^3$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 3y$$
$$\frac{\partial f}{\partial y}(x,y) = -3x - 3y^2$$

stationary points : $(0,0)$, $(-1,1)$

classification: $H_f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & -6y \end{pmatrix}$

at $(-1,1)$: $\begin{pmatrix} -6 & -3 \\ -3 & -6 \end{pmatrix}$ is

negative definite, because leading principal minors are $M_{(1)} = -6 < 0$

$$M_{(2)} = \det(H_f(-1,1))$$
$$= 27 > 0.$$

So $(-1,1)$ is a local maximum.

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at $(0,0)$: $H_f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

is indefinite, because
det $H_f(0,0) \neq 0$ and

$H_f(0,0)$ is not pos.-def. and
not neg def.
and we use (Sylvester)
Thm 306.3).

So $(0,0)$ is a saddle point
and not a local extremum.

(d) Study $f(x,y) = x^3 + y^3 - 3x - 3y$.

VIII 5.2. Conics related to quadratic forms.

Given a quadratic form $q : \mathbb{R}^2 \to \mathbb{R}$ we study the curves given by

$$q(\underline{x}) = k \quad , k \in \mathbb{R} \text{ a}$$

constant. (level curve)

Which curves can occur?

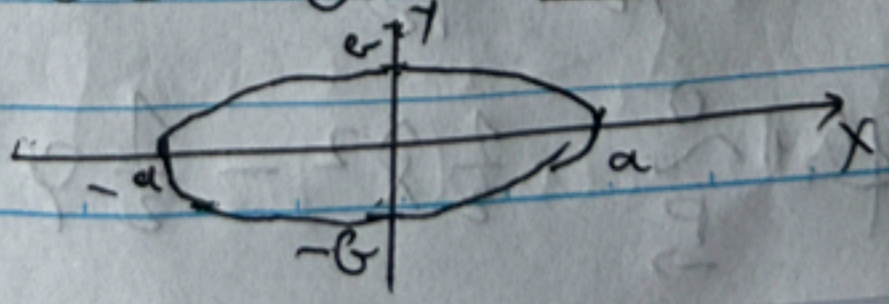
Example 312:

(a) $q(x,y) = 2x^2 + 3y^2$

$$q(x,y) = k \qquad , k > 0$$

$$\iff \frac{x^2}{\sqrt{\frac{k}{2}}} + \left(\frac{y}{\sqrt{\frac{k}{2}}}\right)^2 = 1$$

A curve given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$, $a, b > 0$ constants, is called an <u>ellipse</u>.



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(e) $q(x,y) = x \cdot y \quad , \ (x,y) \in \mathbb{R}^2$

$q = k$

$\underline{k=0}$: curve $= (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$
$= x\text{-axis} \cup y\text{-axis}$

union of two lines

$k \neq 0$: $k > 0$ ($k < 0$ similar)

$$xy = (x,y) \underbrace{\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\mathrm{Eig}(A, \frac{1}{2}) = \mathbb{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\mathrm{Eig}(A, -\frac{1}{2}) = \mathbb{R}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

A is indefinite. (Why?)

$$P := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad P^T A P = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}$$

So $q(x,y) \underset{\overrightarrow{P}}{\overset{\circ}{\sim}} \frac{1}{2}\tilde{x}^2 - \frac{1}{2}\tilde{y}^2$

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under $\binom{x}{y} = P\binom{\tilde{x}}{\tilde{y}}$, $P \in O_2(\mathbb{R})$.

$$\frac{1}{2}\tilde{x}^2 - \frac{1}{2}\tilde{y}^2 = k$$

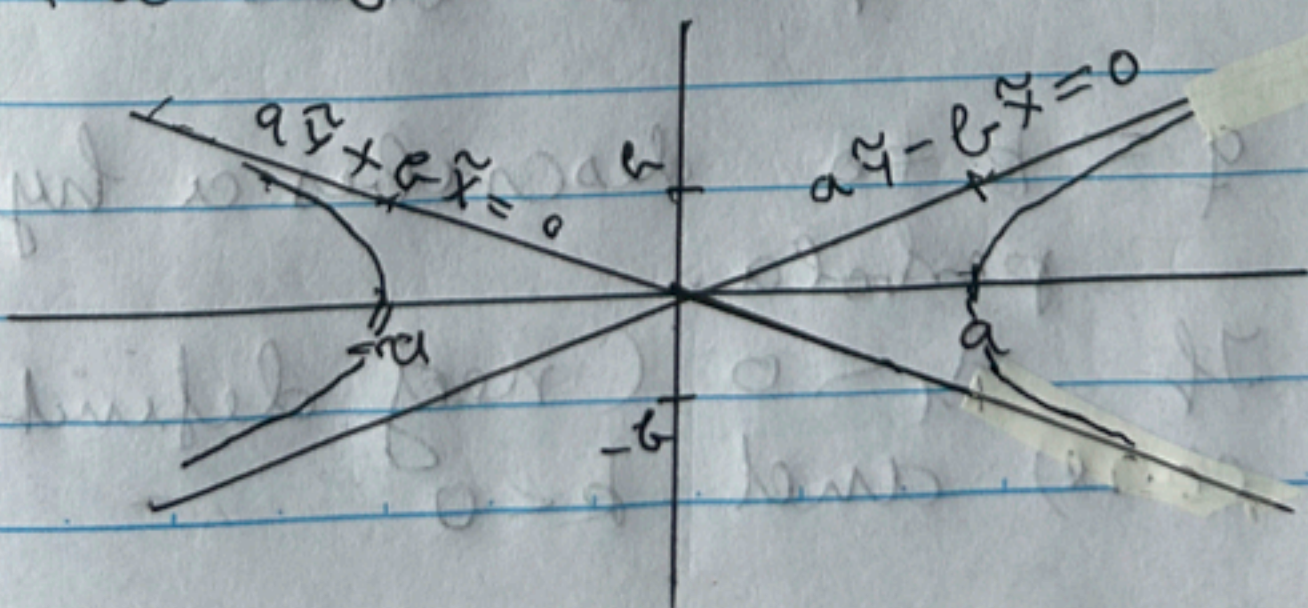
$$\iff \left(\frac{\tilde{x}}{\sqrt{2k}}\right)^2 - \left(\frac{\tilde{y}}{\sqrt{2k}}\right)^2 = 1$$

A curve given by

$$\left(\frac{\tilde{x}}{a}\right)^2 - \left(\frac{\tilde{y}}{b}\right)^2 = 1$$

($a, b > 0$ constants) is called a hyperbola.

$$\left(\frac{\tilde{x}}{a} - \frac{\tilde{y}}{b}\right)\left(\frac{\tilde{x}}{a} + \frac{\tilde{y}}{b}\right) = 1$$



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What happens for $k \downarrow 0$ and $k < 0$?

Theorem 313 Let $q : \mathbb{R}^2 \to \mathbb{R}$ be a quadratic form
$q \simeq d = \lambda_1 \tilde{x}^2 + \lambda_2 \tilde{y}^2$.
with $\lambda_1 \geq \lambda_2$ the eigen values of A ($q = q_A$)

(a) If $\lambda_2 > 0$ (pos. definite case) and $k > 0$ Then
$q = k$ describes an ellipse

(b) If $\lambda_1 > 0 > \lambda_2$ (indefinite case) then
$q = 0$ describes a union of two lines

$q = k \neq 0$ describes a hyperbola.

(c) If $\lambda_1 < 0$ (neg. definite case) and $k < 0$

(semidefinite case)

(d) If $\lambda_1 \cdot \lambda_2 = 0$, then

$$q = 0 \text{ describes } \begin{cases} \mathbb{R}^2, & \text{if } \lambda_1 = \lambda_2 = 0 \\ \text{one line}, & \text{if } \lambda_1 \neq \lambda_2 \end{cases}$$

and for $(\lambda_1 > \lambda_2 = 0, k > 0)$
and $(\lambda_2 < \lambda_1 = 0, k < 0)$
$q = k$ describes the disjoint union of two lines.

(e) In all other cases for $(\lambda_1, \lambda_2, k)$
$q = k$ describes a finite set.

Proof: Exercise $\square$

Example: $q(x,y) = x^2 + 3xy + 2y^2$

$$= \left(x + \frac{3}{2}y\right)^2 - \frac{1}{4}y^2 = \left(x + \frac{7}{4}y\right)\left(x + \frac{5}{4}y\right)$$

$\Rightarrow$ $q(x,y) = 0$ describes a union of two lines intersection only at zero.

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Thus q is indefinite by Theorem 3B', because this case only occurs in (b).