

$$f \text{ injective} \stackrel{320}{\Rightarrow} \ker(f) = \{0_V\} \\ \Rightarrow \text{nullity}(f) = 0$$

$$\stackrel{(*)}{\Rightarrow} \text{rk}(f) = \dim V = \dim W$$

$$\Rightarrow \text{range}(f) = W \Rightarrow f \text{ is surjective}$$

$$\Rightarrow f \text{ is an isomorphism.}$$

$$\uparrow$$

$$f \text{ injective}$$

$$2^0 \Rightarrow 1^0 \quad f \text{ is surjective}$$

$$\Rightarrow \text{range}(f) = W \Rightarrow \text{rk}(f) = \dim W$$

$$\stackrel{(*)}{\Rightarrow} \text{nullity}(f) = 0$$

$$\Rightarrow \ker(f) = \{0_V\} \stackrel{320}{\Rightarrow} f \text{ is in-} \\ \text{jective}$$

$$\stackrel{\uparrow}{\Rightarrow} f \text{ is an isomorphism. } \square$$

$$\uparrow$$

$$\text{surjective}$$

Example 324: (a)

$$f: \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}^{2 \times 1} \quad f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

$$\ker(f) = \text{null}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$\Rightarrow$ f is injective

$\Rightarrow$ f is an isomorphism.

Showing that the kernel is zero is sometimes easier than showing surjectivity.

$$(c) \quad f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 2x_2 - x_1 \\ x_1 + x_2 + 3x_3 \\ x_1 + 2x_3 \end{pmatrix}, \quad f: \mathbb{R}^{3 \times 1} \rightarrow \mathbb{R}^{3 \times 1}$$

$$\text{so } f = T_A, \quad A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\begin{aligned} \ker(f) &= \left\{ v \in \mathbb{R}^{3 \times 1} \mid f(v) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \\ &= \text{null}(A) \\ &= \mathbb{R} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \neq \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

$\Rightarrow f$ is not injective.

Thm 323
 $\Rightarrow$ f is not surjective.VIII 2. Composition of mapsDef 325: (Composition of maps)

Given two maps

$$f: X \rightarrow Y \text{ and } g: Y \rightarrow Z$$

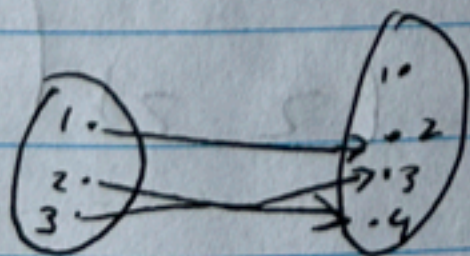
we call the following map
the composition of g with f :

$$(g \circ f)(x) := g(f(x))$$

$$g \circ f: X \rightarrow Z.$$

(Read: $g \circ f$ "g circle f")Example 326: (a) $f: \{1, 2, 3\} \rightarrow \{1, 2, 3, 4\}$

$$f(i) = \begin{cases} 2, & i=1 \\ 4, & i=2 \\ 3, & i=3 \end{cases}$$



$$g: \{1, 2, 3, 4\} \rightarrow \mathbb{R}$$

$$g(i) = i^2$$

$$(g \circ f)(i) = \begin{cases} 4, & i=1 \\ 16, & i=2 \\ 9, & i=3 \end{cases}$$

$$(b) \quad f = T_A : \mathbb{R}^{2 \times 1} \longrightarrow \mathbb{R}^{3 \times 1}$$

$$g = T_B : \mathbb{R}^{3 \times 1} \longrightarrow \mathbb{R}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad B = (1 \ 1 \ 0)$$

$$(b1) \quad (g \circ f)\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = g\left(f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)\right)$$

$$= g\left(A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = g\left(\begin{pmatrix} x_1 + 2x_2 \\ x_1 \\ -x_1 + x_2 \end{pmatrix}\right)$$

$$= B\begin{pmatrix} x_1 + 2x_2 \\ x_1 \\ -x_1 + x_2 \end{pmatrix} = (x_1 + 2x_2) + x_1$$

$$= 2(x_1 + x_2) = (2 \ 2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= T_{(2 \ 2)}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$$(b2) \quad B \circ A = (1 \ 1 \ 0) \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= (2 \ 2)$$

So $T_B \circ T_A = T_{B \circ A}$ is also linear.

Prop 3.27: Let $f: V \rightarrow W$ and $g: W \rightarrow U$ be linear transformations.

Then $g \circ f: V \rightarrow U$ is linear.

$$\begin{aligned} \text{Proof: } (g \circ f)(\lambda v) &= g(f(\lambda v)) \\ &\stackrel{\uparrow}{=} g(\lambda(f(v))) \stackrel{\uparrow}{=} \lambda g(f(v)) \\ &\quad \uparrow \qquad \qquad \uparrow \\ &\quad f \text{ homogeneous} \quad g \text{ homog.} \end{aligned}$$

$$= \lambda (g \circ f)(v)$$

$$\begin{aligned} (g \circ f)(v_1 + v_2) &= g(f(v_1 + v_2)) \\ &\stackrel{\uparrow}{=} g(f(v_1) + f(v_2)) \end{aligned}$$

$\uparrow$
 f additive

$$\stackrel{\uparrow}{=} g(f(v_1)) + g(f(v_2))$$

$\uparrow$
 g additive

$$= (g \circ f)(v_1) + (g \circ f)(v_2) \quad \square$$

Theorem 328: Let $f: V \rightarrow W$

be linear and V, W finite dimensional

⑥ $\text{id}_V: V \rightarrow V$ $\text{id}_V(v) = v$
is linear. (id_V is called the
identity map)

① f is injective $\Leftrightarrow \exists g \in \text{Hom}(W, V): g \circ f = \text{id}_V$
(g is called a "left inverse of f ")

② f is surjective $\Leftrightarrow \exists h \in \text{Hom}(W, V):$
 $f \circ h = \text{id}_W$
(h is called a "right inverse of f ")

③ f is an isomorphism
 $\Leftrightarrow \exists g \in \text{Hom}(W, V): g \circ f = \text{id}_V$ and
 $f \circ g = \text{id}_W$

(In this case g is unique and
denoted by f^{-1} , the "inverse of f ")

Lemma 329 (Linear extension)

Let V and W be vector spaces and $\{v_1, \dots, v_n\}$ be a basis of V and $w_1, \dots, w_n \in W$.

Then $\exists!$ $f \in \text{Hom}(V, W)$ such that $f(v_i) = w_i$, $i = 1, \dots, n$.

Proof: For $v \stackrel{(*)}{=} \sum_{i=1}^n \lambda_i v_i$, $\lambda_i \in \mathbb{R}$
put $f(v) := \sum_{i=1}^n \lambda_i w_i$.

f is well-def., because $(*)$ is unique.

f is linear:

$$\begin{aligned} f\left(\sum \lambda_i v_i + \sum \mu_i v_i\right) &= f\left(\sum (\lambda_i + \mu_i) v_i\right) \\ &= \sum (\lambda_i + \mu_i) w_i = \sum \lambda_i w_i + \sum \mu_i w_i \\ &\stackrel{\text{def}}{=} f\left(\sum \lambda_i v_i\right) + f\left(\sum \mu_i v_i\right) \end{aligned}$$

so f is additive

f is homogeneous (exercise) $\square$

Proof of Theorem 3.28:

(a) Exercise.

(1) " $\Rightarrow$ " Suppose f is injective.

Take a basis $\{v_1, \dots, v_n\}$ of V .

Then $\{f(v_1), \dots, f(v_n)\}$ is linear independent.

$$\text{Pf: } \sum \lambda_i f(v_i) = 0$$

$$\Rightarrow f\left(\sum \lambda_i v_i\right) = 0 \Rightarrow \sum \lambda_i v_i = 0$$

$\uparrow$
 f is injective

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0. \quad \square$$

$\uparrow$
basis

Extend to a basis $\{f(v_1), \dots, f(v_n), w_{n+1}, \dots, w_m\}$

lem. 3.29 $\Rightarrow \exists g \in \text{Hom}(W, V)$:

$$g(f(v_i)) = v_i, \quad i = 1, \dots, n$$

$$\text{and } g(w_j) = 0_V, \quad j = n+1, \dots, m.$$

" $\Leftarrow$ " $\exists g \in \text{Hom}(W, V)$: $g \circ f = \text{id}_V$

To show f is injective

$$f(v) = 0 \Rightarrow v = \text{id}_V(v) = (g \circ f)(v) \\ = g(f(v)) = g(0_W) = 0_V. \Rightarrow \ker(f) = \{0_V\}.$$

② Exercise.

③ f bijective $\Leftrightarrow f$ inj and f surj.

$$\stackrel{1), 2)}{\Leftrightarrow} \exists g, h \in \text{Hom}(W, V) : g \circ f = \text{id}_V \\ \text{and } f \circ h = \text{id}_W$$

To show $g = h$.

$$\begin{aligned} h(w) &= (\text{id}_V \circ h)(w) \\ &= g(f(h(w))) = g(\text{id}_W(w)) \\ &= g(w). \quad \square \end{aligned}$$

Example 330: (a) $f: \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}, f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = x_1 + x_2$

is surjective.

Find right inverse $h: \mathbb{R} \rightarrow \mathbb{R}^{2 \times 1}$

$\{1\}$ is a basis of $\mathbb{R}$.

$$f(\vec{e}_1) = 1.$$

Take $h \in \text{Hom}(\mathbb{R}, \mathbb{R}^{2 \times 1})$ s.t.

$$h(1) = \vec{e}_1.$$

$$h(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

$$(f \circ h)(x) = x + 0 = x.$$

$$(b) f: \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}^{3 \times 1}$$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_1 \\ x_1 \end{pmatrix} = T_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \ker(f) = \text{null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$\Rightarrow f$ injective

Find left inverse:

$$f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{Take } g \text{ with } g\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$g\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$g\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(Lemma 3.29)

$$g \in \text{Hom}(\mathbb{R}^{3 \times 1}, \mathbb{R}^{2 \times 1})$$

$$\text{Thus } \exists B \in \mathbb{R}^{2 \times 3} : g = T_B.$$

Which B?

$$B \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{\text{column alg.}} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \} B$$

$$(c) \quad f: \mathbb{R}^{3 \times 1} \rightarrow \mathbb{R}^{3 \times 1}$$

$$f = T_A \quad A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\ker(f) = \text{null}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\Rightarrow$ f is an isomorphism.

$$f^{-1} = T_{A^{-1}} \quad A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 2 & -1 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

VTB 3. Matrix of a linear transformation w.r.t. to bases.

We understand $f \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ through a matrix. $f = T_A$

But sometimes A is not beautiful.

$\leadsto$ change the bases.

Def 331! Let $f \in \text{Hom}(V, W)$
and

$B = \{v_1, \dots, v_n\}$ be a basis of V

$C = \{w_1, \dots, w_m\}$ — a — W

Matrix of f w.r.t. B and C
(from B to C):

$$[f]_{B \rightarrow C} := [f]_{C \leftarrow B}$$

$$:= \begin{pmatrix} [f(v_1)]_C & [f(v_2)]_C & \dots & [f(v_n)]_C \end{pmatrix}$$

if we write $f(v_1), \dots, f(v_n)$ wrt. C and store the coordinates.

Example 332:

$$(a) \quad f: \mathbb{R}^{2 \times 1} \longrightarrow \mathbb{R}^{3 \times 1}$$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 - x_1 \\ 2x_2 + x_1 \end{pmatrix}$$

$$f = T_A, \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[f]_{\mathcal{B} \rightarrow \mathcal{C}} = [f]_{\mathcal{C} \rightarrow \mathcal{C}}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= A$$

$$\left(f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1 \vec{e}_1 - \vec{e}_2 + \vec{e}_3 \right)$$

$$[f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)]_{\mathcal{C}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \quad [f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = A\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = (-3)\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\text{So } \left[f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right]_C = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$$

$$f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = A\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\left[f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)\right]_C = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{So } [f]_{C \leftarrow B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \\ 1 & 1 \end{pmatrix}$$

(e) Other way for $[f]_{C \leftarrow B}$.

$$[f]_{C \leftarrow B} = P_{C \leftarrow M} [f]_{M \leftarrow M} P_{M \leftarrow B}$$

$$P_{M \leftarrow B} = [\text{id}_{\mathbb{R}^2}]_{M \leftarrow B}$$

$$= \left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_M, \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]_M \right)$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$P_{C \leftarrow M} = (P_{M \leftarrow C})^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$[f]_{C \leftarrow B} = \frac{1}{2} \begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -2 & -4 \\ 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \\ 1 & 1 \end{pmatrix}$$

$$(c) \quad P_2 := R_0 + R_1 + R_2$$

$$P_4 := R_0 + R_1 + \dots + R_4$$

$$P_n(x) = x^n \quad P_i \in \text{Poly}(\mathbb{R})$$

$g: P_2 \rightarrow P_4$ defined via

$$g(p) = q$$

with $q(x) := p(x^2+1), p \in P_2$

(c1) g is linear

$$(c2) \quad [g]$$

$$\{p_0, p_1, p_2\} \rightarrow \{p_0, p_1, p_2, p_3, p_4\}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Given $V \xrightarrow{f} V \xrightarrow{g} V$

(think dim vector spaces)

and bases B, C, D .

Then $v \in V \xrightarrow{f} W \xrightarrow{g} U$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \\ [v]_B & \mathbb{R}^{n \times 1} & \xrightarrow{[f]} & \mathbb{R}^{m \times 1} & \xrightarrow{[g]} & \mathbb{R}^{l \times 1} \\ & & [f]_{B \rightarrow C} & & [g]_{C \rightarrow D} \end{array}$$

$$[g \circ f]_{D \leftarrow B} = [g]_{D \leftarrow C} \circ [f]_{C \leftarrow B}$$

Ex: $\mathbb{R}^{2 \times 1} \xrightarrow{f} P_2 \xrightarrow{g} P_4$

$$f\left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \underbrace{(t_1 + t_2)}_{\text{real number}} \underbrace{P_0}_{\text{polynomial function}} + (t_2 - t_1)P_1 + (2t_2 + t_1)P_2$$

$(g(P))(x) := P(x^2 + 1)$
map from (C) .

$$B := \{(1), (-1)\}, \quad C = \{P_0 + P_1 - P_2, P_0 + P_1, P_0 - P_1\}$$

$$D = \{P_0, P_1, P_2, P_3, P_4\}$$

Direct way of computing $D \rightarrow B$ $[g \circ f]$

$$(g \circ f) \left(\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right) (x) = (t_1 + t_2) p(x)$$

$$+ (t_2 - t_1) p(x^2 + 1)$$

$$+ (2t_2 + t_1) p_2(x^2 + 1)$$

$$= t_1 + t_2 + (t_2 - t_1)(x^2 + 1)$$

$$+ (2t_2 + t_1)(x^4 + 2x^2 + 1)$$

$$= (4t_2 + t_1) + (5t_2 + t_1)x^2$$

$$+ (2t_2 + t_1)x^4$$

$$(g \circ f) \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) (x) = 5 + 6x^2 + 3x^4$$

$$(g \circ f) \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) (x) = -3 - 4x^2 - x^4$$

$$So \quad [g \circ f]_{D \rightarrow B} = \begin{pmatrix} 5 & 6 & 3 \\ -3 & -4 & -1 \end{pmatrix}$$

Way using compositions of matrices

$$[g \circ f]_{D \rightarrow B} = [g]_{D \rightarrow C} [f]_{C \rightarrow B}$$

$$= \begin{pmatrix} 5 & -3 & 0 & 0 & 6 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $[g \circ f] = [g] \circ [f]$ $D \leftarrow B$ $D \leftarrow C$ $C \leftarrow B$

$$= \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$[g] \circ [f] = [g] \circ [f]$ $D \leftarrow \{e_1, e_2\}$ $D \leftarrow C$ $C \leftarrow B$

We have $[f] = \begin{pmatrix} -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

Date

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Remark 333: (polynomial functions and polynomials)

polynomial function: $p: \mathbb{R} \rightarrow \mathbb{R}$, $p(t) = \sum_{i=0}^n a_i \cdot t^i$

Ex: $p: \mathbb{R} \rightarrow \mathbb{R}$, $p(t) = 2t^2 + t - 1$

polynomial: expression of the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Ex: $2x^2 + x - 1$

A polynomial $Q(x)$ defines a polynomial function $\mathbb{R} \rightarrow \mathbb{R}$ $x \mapsto Q(x)$

Ex: The polynomial $Q(x) = x^2 + 2x - 3$ defines $q: \mathbb{R} \rightarrow \mathbb{R}$ $q(t) := Q(t) = t^2 + 2t - 3$

Local pol

Look at Example 332 (c) in terms of polynomials.

$$P_2 = \mathbb{R}X^0 + \mathbb{R}X + \mathbb{R}X^2 \supseteq \{1, X, X^2\} \text{ basis}$$

$$P_4 = \mathbb{R}X^0 + \mathbb{R}X + \mathbb{R}X^2 + \mathbb{R}X^3 + \mathbb{R}X^4 \supseteq \{1, X, X^2, X^3, X^4\} \text{ basis}$$

$$g: P_2 \rightarrow P_4 \quad g(Q(X)) := Q(X^2+1)$$

$$\text{e.g. } g(X^2+2) = (X^2+1)^2+2$$

$$= X^4+2X^2+3$$

$$[g] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{\{1, X, X^2\} \rightarrow \{1, X, X^2, X^3, X^4\}}$$

In Example 332 (d)

$$[g \circ f] = \begin{pmatrix} 5 & 0 & -3 \\ 6 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & -1 \end{pmatrix} \xrightarrow{\{1, X, X^2, X^3\} \rightarrow \{1, X, X^2, X^3, X^4\}}$$