

Chapter VIII

Linear Transformations

VIII.1. 1st Definitions

We have already defined linear maps in Def 1.97.

This chapter is devoted to study those map thoroughly.

Motivation: 1) Most vector spaces are not n -spaces, because

- They are infinite dimensional or
- They have no canonical choice of a basis.

2) The matrix $A \in \mathbb{R}^{m \times n}$ defining $T_A: \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}^{m \times 1}$, $T_A(v) = Av$ may not be the "nicest" matrix to describe the map.

So it is better to have a coordinate independent concept of linear map.

Def 314: Let V, W be vector spaces.

1) A map $f: V \rightarrow W$ is called linear if

$$\cdot \forall \lambda \in \mathbb{R} \forall v \in V: f(\lambda v) = \lambda f(v) \quad \text{and} \\ \text{(homogeneous)}$$

$$\cdot \forall v_1, v_2 \in V: f(v_1 + v_2) = f(v_1) + f(v_2) \\ \text{(additive)}$$

A linear map is also called a linear transformation

or homomorphism

2) A linear transformation $f: V \rightarrow W$

is called a monomorphism

if f is injective (one-to-one)

(Recall that means

For $v_1 \neq v_2$ in V we have $f(v_1) \neq f(v_2)$.)

3) A linear transformation is called

epimorphism if f is surjective (onto)

(Recall that means

$$\forall w \in W \exists v \in V: f(v) = w)$$

4) A linear transformation $f: V \rightarrow W$ is called isomorphism if f is bijective, i.e. f is surjective and injective.

Example 315: (see also Example 321)

1) $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = 2x$

$$f(x_1 + x_2) = 2(x_1 + x_2) = 2x_1 + 2x_2 = f(x_1) + f(x_2)$$

$$f(\lambda x) = 2(\lambda x) = (\lambda 2x) = \lambda(2x) = \lambda f(x).$$

So f is additive and homogeneous
Thus f is linear.

2) $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^2$

Then $f(1+1) = f(2) = 4$

$$f(1) + f(1) = 1^2 + 1^2 = 2 \neq 4.$$

So f is not additive $\Rightarrow f$ is not linear.

3) $f: \mathbb{R}^{3 \times 1} \rightarrow \mathbb{R}^{2 \times 1}$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - 2x_2 \\ x_3 - x_1 + x_2 \end{pmatrix}$$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}\right) = \longrightarrow$$

$$\begin{aligned}
 & \begin{pmatrix} (x_1 + y_1) - 2(x_2 + y_2) \\ (x_3 + y_3) - (x_1 + y_1) + (x_2 + y_2) \end{pmatrix} \\
 &= \begin{pmatrix} x_1 - 2x_2 \\ x_3 - x_1 + x_2 \end{pmatrix} + \begin{pmatrix} y_1 - 2y_2 \\ y_3 - y_1 + y_2 \end{pmatrix} \\
 &= f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) + f\left(\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right)
 \end{aligned}$$

so f is additive.

$$f\left(\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = f\left(\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \lambda x_1 - 2(\lambda x_2) \\ \lambda x_3 - \lambda x_1 + \lambda x_2 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} x_1 - 2x_2 \\ x_3 - x_1 + x_2 \end{pmatrix} = \lambda f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right).$$

So f is homogeneous.

Thus f is linear.

$$4) f: C^1(\mathbb{R}) \rightarrow C(\mathbb{R})$$

$$f(\varphi) := \varphi' \quad \text{derivation}$$

$$\begin{aligned}
 f(\varphi_1 + \varphi_2) &= (\varphi_1 + \varphi_2)' = \varphi_1' + \varphi_2' \\
 &= f(\varphi_1) + f(\varphi_2)
 \end{aligned}$$

$$\bullet \quad f(\lambda \varphi) = (\lambda \varphi)' = \lambda \varphi' = \lambda f(\varphi)$$

for $\lambda \in \mathbb{R}$, $\varphi \in C^1(\mathbb{R})$.

$$5) \quad f: C(\mathbb{R}) \rightarrow C^1(\mathbb{R})$$

$$f(\varphi) := \int_0^x \varphi(t) dt$$

$$x \mapsto \int_0^x \varphi(t) dt$$

$$(f(\varphi + \psi))(x) = \int_0^x \varphi(t) + \psi(t) dt$$

$$= \int_0^x \varphi(t) dt + \int_0^x \psi(t) dt$$

$$= (f(\varphi))(x) + (f(\psi))(x)$$

$$= (f(\varphi) + f(\psi))(x)$$

$\uparrow$

addition

in $C^1(\mathbb{R})$

$f(\lambda \varphi) = \lambda f(\varphi)$. So f is linear.

Example 316: 1) $f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) = 2x$

injective? $x_1 \neq x_2 \Rightarrow f(x_1) = 2x_1$

$$f(x_2) = 2x_2$$

$\Rightarrow f$ is injective.

Better way to show injectivity:

$$\forall x_1, x_2 : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$$

Take $x_1, x_2 \in \mathbb{R}$. Suppose $f(x_1) = f(x_2)$

$$\Rightarrow 2x_1 = 2x_2 \Rightarrow 2(x_1 - x_2) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

$\uparrow$
 $\cdot \frac{1}{2}$

So f is injective.

surjective? Take $y \in \mathbb{R}$.

We need to find $x \in \mathbb{R}$ such that $f(x) = y$.

$$y = f(x) = 2x \Rightarrow x = \frac{y}{2}$$

Take $x = \frac{y}{2}$.

$$\text{Then } f(x) = 2\left(\frac{y}{2}\right) = y.$$

So f is surjective.

bijective? yes, because injective and surjective.

2) $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = x^2$

injective?

No!

$f(1) = 1 = f(-1)$

surjective?

No!

$-1 = f(x) = x^2$

has no solution in $\mathbb{R}$.

3) $f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$f: \mathbb{R}^{3 \times 1} \rightarrow \mathbb{R}^{2 \times 1}$

injective? No! $f\left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = f\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$

surjective? Take $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^{2 \times 1}$

Then $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = -y_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} - y_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= f\left(\begin{pmatrix} -y_1 \\ -y_1 \\ y_2 \end{pmatrix}\right)$

So f is surjective.

bijjective? No! because not injective.

$$4) f: C'(\mathbb{R}) \rightarrow C(\mathbb{R}) \quad f(\varphi) = \varphi'$$

injective? No! $f(1) = 0 = f(0)$
constant maps

surjective? Yes! Take $\varphi \in C(\mathbb{R})$

$$f\left(\int_0^x \varphi(t) dt\right) = \left(\int_0^x \varphi(t) dt\right)' = \varphi$$

$$5) \text{ Study } f: C(\mathbb{R}) \rightarrow C'(\mathbb{R})$$

$$f(\varphi) = \int_0^x \varphi(t) dt.$$

wrt. injectivity and surjectivity.

Prop. 3/7: 1) For $A \in \mathbb{R}^{m \times n}$ the
map $T_A: \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}^{m \times 1}$ $T_A(v) := Av$
is linear

2) Let $f: \mathbb{R}^{n \times 1} \rightarrow \mathbb{R}^{m \times 1}$ be linear.
Then $\exists A \in \mathbb{R}^{m \times n}: f = T_A.$

Proof:

$$\begin{aligned}
 1) \quad T_A(v_1 + v_2) &= A(v_1 + v_2) \\
 &= Av_1 + Av_2 = T_A(v_1) + T_A(v_2) \\
 T_A(\lambda v) &= A(\lambda v) = \lambda(Av) = \\
 &= \lambda T_A(v)
 \end{aligned}$$

2) We are given a linear transf.
 $f: \mathbb{R}^{n \times 1} \longrightarrow \mathbb{R}^{m \times 1}$.

Consider $\begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} = f(\vec{e}_i)$

Put $A := \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$

Claim $f = T_A$.

Proof: We know that f and T_A are linear.

Exercise: We only need to show that f and T_A coincide on generating set.

Take $S = \{ \vec{e}_1, \dots, \vec{e}_n \}$

$$f(\vec{e}_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix} = A \vec{e}_i = T_A(\vec{e}_i) \checkmark$$

for $i=1, \dots, n$.

S generates $\mathbb{R}^{n \times 1}$. Thus $f = T_A$ $\square$ (Claim)

$\square$

(Prop 317)

Thus for $\mathbb{R}^n$ we don't get something new, but it is still helpful, because A could be not very beautiful.

Example 318: (a) $f: \mathbb{R}^{2 \times 1} \longrightarrow \mathbb{R}^{2 \times 1}$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix}$$

Find A !

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So $f = T_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$.

$$(4) f: \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}^{2 \times 1}$$

orthogonal projection onto $\mathbb{R} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Find A , s.t. $f = T_A$.

$$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(because $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \end{pmatrix}$)

Want $A \in \mathbb{R}^{2 \times 2}$ such that

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ i.e.}$$

$$A \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\text{Invert } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}: \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\text{So } A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\text{So } f = T_{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}$$

$$\text{Check: } \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \checkmark$$

Def 319: 1) Let V, W be vector spaces
 $\text{Hom}(V, W) := \{f: V \rightarrow W \mid f \text{ linear}\}$

2) Take $f \in \text{Hom}(V, W)$.

$$\ker(f) := \{v \in V \mid f(v) = 0\}$$

kernel of f

$$\text{range}(f) := \{f(v) \mid v \in V\}$$

range of f
 (also image of f)

Theorem 320: Let $f \in \text{Hom}(V, W)$.

1) $\ker(f)$ and $\text{range}(f)$
 are subspaces.

2) f injective $\Leftrightarrow \ker(f) = \{0\}$

3) f surjective $\Leftrightarrow \exists S \subseteq W$:
 S generates W and $S \subseteq \text{range}(f)$.

We write $\text{nullity}(f) := \dim \ker(f)$
 $\text{rank}(f) = \text{rk}(f) := \dim \text{range}(f)$

Proof: 1) Just for $\ker(f)$.

To show $\ker(f) \leq V$. f linear

$$0 \in \ker(f) : f(0) = f(0+0) \stackrel{f \text{ linear}}{=} f(0) + f(0)$$

$$\Rightarrow -f(0) \quad 0 = f(0) \Rightarrow 0_v \in \ker(f)$$

closed under $+$: $v_1, v_2 \in \ker(f)$.

$$f(v_1 + v_2) = f(v_1) + f(v_2) = 0_v + 0_v = 0_v$$

$$\Rightarrow v_1 + v_2 \in \ker(f)$$

closed under $\cdot : \mathbb{R} \times V \rightarrow V : \lambda \in \mathbb{R}, v \in \ker(f)$

$$f(\lambda v) = \lambda f(v) = \lambda \cdot 0_v = 0_v$$

$$\Rightarrow \lambda v \in \ker(f)$$

Subspace criterion $\Rightarrow \ker(f) \leq V$.

2) " $\Rightarrow$ " For $v \in \ker(f)$ we have

$$f(v) = 0_v = f(0_v)$$

$$f \text{ injective} \Rightarrow v = 0_v$$

$$\text{Thus } \ker(f) = \{0_v\}.$$

" $\Leftarrow$ " We have $\ker(f) = \{0_V\}$
 To show f is injective.

$$f(v_1) = f(v_2) \Rightarrow f(v_1) - f(v_2) = 0_W$$

$f \text{ linear} \longrightarrow "$

$$f(v_1 - v_2)$$

$$\Rightarrow v_1 - v_2 \in \ker(f) = \{0_V\}$$

$$\Rightarrow v_1 - v_2 = 0_V \Rightarrow v_1 = v_2$$

3) " $\Rightarrow$ " f surjective $\Rightarrow \text{range}(f) = W$

Take S to be W .

" $\Leftarrow$ " Suppose $\exists S$ generating W :
 $\text{range}(f) \supseteq S$.

$$\Rightarrow \text{range}(f) \supseteq \text{span}(S) = W.$$

$\uparrow$
 subspace
 of W

$\Rightarrow f$ is surjective

$\square$

(for 315)

Example 321:Let $f: \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}^{3 \times 1}$
be linear and

$$f\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \text{ and } f\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

What is $f\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right)$?

$$\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{7}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{9}{5} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -5 & -9 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cc|c} 1 & 0 & \frac{7}{5} \\ 0 & 1 & \frac{9}{5} \end{array} \right) \quad \begin{array}{l} \lambda_1 = \frac{7}{5} \\ \lambda_2 = \frac{9}{5} \end{array}$$

$$\text{Then } f\left(\begin{pmatrix} 5 \\ 1 \end{pmatrix}\right) = f\left(\frac{7}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{9}{5} \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$$

$$\stackrel{\uparrow}{=} f\left(\frac{7}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + f\left(\frac{9}{5} \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$$

linear

$$\stackrel{\downarrow}{=} \frac{7}{5} f\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) + \frac{9}{5} f\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)$$

$$= \frac{7}{5} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \frac{9}{5} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{16}{5} \\ \frac{2}{5} \\ \frac{39}{5} \end{pmatrix}$$

Example 322:

$$(a) \quad f: \mathbb{R} \rightarrow \mathbb{R}^2 \quad f(x) = \begin{pmatrix} x \\ 2x \end{pmatrix}.$$

f is linear, because $f = T_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$.

injective? $\ker(f) = \{x \in \mathbb{R} \mid f(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$

$$= \{x \in \mathbb{R} \mid \begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\} = \{0_{\mathbb{R}}\} \quad (b)$$

So f is injective by Thm 320
surjective? No! $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \text{range}(f)$.

$$(b) \quad f: \mathbb{R}^{3 \times 1} \rightarrow \mathbb{R}^{3 \times 1}$$

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 + x_2 + 2x_3 \\ -x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

$$f = T_A \quad \text{for } A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

so is linear

$$\ker(f) = \text{null}(A) = \mathbb{R} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

So f is not injective.

surjective?

$\text{range}(f) = \text{col}(A)$ has dimension

(by dim theorem) $3 - \text{nullity}(A) = 3 - 1 = 2 \neq 3$

So f is not surjective.

(c) $f: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$

$$(a_n)_{n \in \mathbb{N}} \mapsto (a_{n+1})_{n \in \mathbb{N}}$$

f is linear (Exercise)

f is not injective: $f((1, 0, 0, \dots)) = (0, 0, \dots)$

f is surjective: Take $(b_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$

Put $a_1 = 0$, $a_n = b_{n-1}$ for $n \geq 1$.
 $\Rightarrow f((a_n)) = (b_n)_{n \in \mathbb{N}}$.

(d) Find $f: V \rightarrow V$ linear which is injective, but not surjective.

— End of Lecture 10th of Jan 2024

Theorem 323: Let V, W be

finite dimensional vector spaces and f linear and $\dim V = \dim W$.

T.a.e:

1^o f is bijective
(an isomorphism)

2^o f is surjective

3^o f is injective.

Proof: $1^o \Rightarrow 2^o \checkmark$ and $1^o \Rightarrow 3^o \checkmark$
by definition of bijectiveness.

$3^o \Rightarrow 1^o$ By the dimension theorem (204) we have

$$(*) \quad \text{rk}(f) + \text{nullity}(f) = \dim V$$