LINEAR ALGEBRA 1
PROBLEM SHEET 14

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Problem 1 (10+10+10, matrix of an linear map w.r.t a pair of bases)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that $f(2,1) = (0,1,1)$ and $f(-1,2) = (2,3,2)$.

(i) Compute $f(1,1)$.
(ii) Compute $A \in \mathbb{R}^{3\times 2}$ such that $f = T_A$ (after identifying $\mathbb{R}^2$ with $\mathbb{R}^{2\times 1}$ and $\mathbb{R}^3$ with $\mathbb{R}^{3\times 1}$).
(iii) Compute $[f]_{C\to B}$ for the bases

$$B := \{(1,1,1),(1,-1,1),(-1,0,1)\},\ C := \{(2,1),(-1,2)\}$$

Problem 2 (10+10+10, range)**.** (i) Let $f : V \to W$ be a linear transformation. Prove that range(f) is a subspace of W.
(ii) Given $A \in \mathbb{R}^{m\times n}$. Prove range$(T_A) = \mathrm{col}(A)$.
(iii) Given $f : C^1(\mathbb{R}) \to C(\mathbb{R})$, $f(\varphi) := \varphi' + \varphi$. Compute its kernel and its range.

Problem 3 (10+10+10, hom space)**.** (i) Given vector spaces V and W prove that the set of linear transformation from V to W, denoted by Hom(V, W), is a vector space with respect to

$$(f+g)(v) := f(v) + g(v),\ (\lambda f)(v) := \lambda f(v),\ \lambda \in \mathbb{R},\ v \in V.$$

(ii) Consider the map

$$\Phi : \mathbb{R}^{m\times n} \to \mathrm{Hom}(\mathbb{R}^{n\times 1}, \mathbb{R}^{m\times 1}),\ \Phi(A) := T_A$$

Prove that Φ is an isomorphism.
(iii) Find a basis of $\mathrm{Hom}(\mathbb{R}^{n\times 1}, \mathbb{R}^{m\times 1})$.

Problem 4 (30, linear extension)**.** Determine if there is a linear map satisfying the given conditions.

(i) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(1,1,0) = (1,0,0)$, $f(1,2,1) = (1,0,1)$, $f(3,4,1) = (3,0,1)$,
(ii) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(1,1,2) = (1,0)$, $f(1,1,1) = (0,1)$, $f(6,6,9) = (6,7)$,
(iii) $f : \mathbb{R}^4 \to \mathbb{R}^3$, $f(1,1,0,1) = (1,1,0)$, $f(1,2,1,1) = (1,0,1)$, $f(1,3,4,1) = (3,0,1)$,

Problem 5 (55, left inverse, right inverse)**.** Detect which map is an epimorphism, monomorphism or an isomorphism. If so find a left inverse, a right inverse and an inverse, respectively.

(i) $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1,x_2,x_3) := (x_1 + x_2, x_2 + x_3)$
(ii) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1,x_2,x_3) := (x_1 + 2x_2 + x_3, x_1 - x_2 + x_3, 2x_1 + x_2 + x_3)$
(iii) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x_1,x_2,x_3) := (x_1 + x_2 + 5x_3, x_1 - x_2 + x_3, x_1 + x_2 + 5x_3)$
(iv) $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f(x_1,x_2,x_3,x_4) := (x_1 + x_2 + x_3, x_1 - x_2 + x_3, x_2 + x_3, x_1 + x_3)$.

Date: Please hand in online to your TA by **15th of January 2023 11:59 am**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.