LINEAR ALGEBRA 1
PROBLEM SHEET 13

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Problem 1 (5+5+10+10*+10, Singular value decomposition (SVD))**.** Find a SVD for the following matrices:

(i) $(1\ 2)$
(ii) $(1\ 2\ 3)^T$
(iii) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$
(iv) $\begin{pmatrix} 1 & -\frac{1}{5} & \frac{7}{5} \\ -1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & \frac{11}{5} & -\frac{2}{5} \end{pmatrix}$
(v) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

Problem 2 (15, orthogonal matrices)**.** Prove Proposition 284 (Page 337)

Problem 3 (5+10+(5+15*)+12, quadratic forms)**.** Determine if the following given two quadratic forms are equivalent and if they are orthogonally equivalent.

(i) $x_1^2 + x_2^2$ and $x_1^2 + 2x_1x_2 + 2x_2^2$ on $\mathbb{R}^2$,
(ii) $x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_3$ and $2x_1^2 + 2x_2^2$ on $\mathbb{R}^3$,
(iii) $x_1^2 - x_2^2 - x_3^2 - x_4^2$ and $x_1^2 + x_2^2 + x_3^2 - x_4^2$ on $\mathbb{R}^4$.

Determine the definiteness of the 6 given quadratic forms.

Problem 4 (10+10, extremum on the 1-sphere)**.** Find an orthogonal change of variables which transforms the following quadratic form into a diagonal form. (orthogonal elimination of cross product terms)

$$x_1^2 - x_2^2 + x_3^2 + 2(x_1x_2 - x_1x_3 + x_2x_3)$$

on $\mathbb{R}^3$. What is the maximum value and the minimum value of the form on the unit sphere?

Problem 5 ((10+15+10)+10+20*, Sylvester)**.** (i) Determine the definiteness of the following symmetric matrices.

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} c & a & b \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} (a, b, c \in \mathbb{R}), \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -3 \end{pmatrix}$$

(ii) Show that a positive definite symmetric matrix must have positive determinant.
(iii) Prove Theorem 306(1). *Hint: Induction on the size of the matrix.*

Problem 6 (10*, endomorphisms)**.** Find a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ wich has $(1,1,1)^T$ in its kernel and $(1,1,-1)^T, (1,1,2)^T$ in its range.

Date: Please hand in before the lecture by **10th of January 2024**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.