LINEAR ALGEBRA 1
PROBLEM SHEET 12

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Problem 1 (30, Gram-Schmidt)**.** Gram-Schmidt orthonormalize the column sets of the following matrices to obtain an orthonormal basis for the column space. Further provide a QR decomposition of the matrix.

(i) $\begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$

(iii) $\begin{pmatrix} -1 & 1 & -2 \\ 2 & 2 & -2 \\ -2 & 1 & 1 \\ -1 & -2 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Problem 2 (15+15*, least square regression)**.** Suppose we have observed the following data.

a_i	1	−1	2	3	−2	−3	0
b_i	1.5	1	4.5	8	−2	−4	0.5

(i) Using the least square method, find the affine regression line r (of the form $r(a) = b$) for the above data.

(ii) Minimize $\|Ax - b\|$ over $x \in \mathbb{R}^3$ for the matrix A in 12.1.(iii) and $b = (1,1,1,1,1)^T$. Provide the minimum value and the minimum x.

Problem 3 (15, norm)**.** Let $(V, \langle,\rangle)$ be an inner product space. Prove that $\| \ \| : V \to \mathbb{R}$, given by $\|v\| := \sqrt{\langle v, v\rangle}$, is a norm on V. (See Theorem 123 for the definition of a norm.)

Problem 4 (10+15, projection)**.** Let A be a real $m \times n$ matrix of rank n and let W be its column space. Prove that the orthogonal projection onto W satisfies

$$\mathrm{proj}_W(v) := A(A^TA)^{-1}A^Tv, \ v \in \mathbb{R}^m.$$

Find $\mathrm{proj}_W(e_1)$ for the matrices in 12.1. (Hint: There are different ways to compute the projection map, you have already the QR-decomposition.)

Problem 5 (40, Linear transformations (very easy))**.**

(i) Let $f : V \to W$ and $g : W \to U$ be linear maps. Show that $g \circ f : V \to U$ is a linear transformation. (Recall $(g \circ f)(v)$ is defined to be $g(f(v))$.)

(ii) Consider $f : \mathbb{R}^3 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) := (2x_1 + 3x_3, 2x_3 - x_2, -x_3 - x_2 - 2x_1, 4x_1 + 3x_2)$. Prove that f is linear. Compute the kernel and the range of f.

(iii) Consider $g : C[0,1] \to \mathbb{R}^3$, $g(\varphi) := (\varphi(0), \varphi(0.5), \varphi(1))$. Prove that g is a linear transformation.

(iv) Let $W := \{\varphi \in C[0,1] \mid \varphi(0) = \varphi(1)\}$. Let U be $g(W)$. Prove that $f|_U$ is injective, i.e. that the restriction of f to the subspace U is injective. (Hint: Show that $U \cap \ker(f) = \{0\}$.)

Date: Please hand in before the lecture by **3rd of January 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.