V4. Additional Note: Proof of Theorem 235.

Theorem 235: Let $A \in \mathbb{C}^{n\times n}$, $p_A(\lambda) = (\lambda-\lambda_1)^{\nu_1}\cdots(\lambda-\lambda_r)^{\nu_r}$ with $\lambda_1, \dots, \lambda_r$ distinct.
Then $\nu_i = \dim_{\mathbb{C}} \mathrm{Eig}^{\nu_i}(A, \lambda_i)$ for all $i = 1, \dots, r$.

Example 244:

$$A = \begin{pmatrix} 2 & 1 & & & & & & \\ & 2 & & & & & & \\ & & 2 & 1 & & & & \\ & & & 2 & 1 & & & \\ & & & & 2 & & & \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 \\ & & & & & & & -1 \end{pmatrix}$$

Then $\mathrm{Eig}^5(A, 2) = \mathbb{R}\vec{e}_1 + \dots + \mathbb{R}\vec{e}_5$

$\mathrm{Eig}^3(A, -1) = \mathbb{R}\vec{e}_6 + \mathbb{R}\vec{e}_7 + \mathbb{R}\vec{e}_8$.

The proof needs several steps.

Let $\mathcal{D}_n$ be the set of matrices $\in \mathbb{C}^{n\times n}$ which are diagonalizable over $\mathbb{C}$.

Prop 245: $\mathcal{D}_n$ is dense in $\mathbb{C}^{n\times n}$, i.e.
$\forall A \in \mathbb{C}^{n\times n}$ $\exists$ $B_1, B_2, B_3, \dots \in \mathcal{D}_n$: $B_i \xrightarrow{i\to\infty} A$
wrt. $\|\cdot\|_{Eu}: \mathbb{C}^{n\times n} \to \mathbb{R}$, $\|A\|_{Eu} := \sqrt{\sum_{i,j} |a_{ij}|^2}$

Proof: Induction on n:
$n=1$: ✓, because all elements of $\mathbb{C}^{1\times 1}$ are diagonal.

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<u>$n>1$:</u> Take $\lambda \in \mathrm{Spec}(A)$. and $v_1 \in \mathrm{Eig}_{\mathbb{C}}(A,\lambda) \setminus \{0\}$

We extend $\{v_1\}$ to a basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$.

Put $P = (v_1, v_2, v_3, \dots, v_n) \in \mathbb{C}^{n \times n}$

Then $AP = PB$ with B of the form
$$B = \left(\begin{array}{c|ccc} \lambda & b_{12} & \cdots & b_{1n} \\ \hline 0 & & & \\ \vdots & & C & \\ 0 & & & \end{array}\right)$$

By induction hypothesis

$\exists\ C_1, C_2, C_3, \dots \in \mathcal{D}_{n-1} : C_i \xrightarrow{i \to \infty} C$

w.r.t. $\|\cdot\|_{\mathrm{Eu},n-1}$

Take a sequence $(\mu_j)_{j \in \mathbb{N}}$ in $\mathbb{C}$ s.t. $\mu_j \xrightarrow{j \to \infty} \lambda$ w.r.t. $|\cdot|_{\mathbb{C}}$ and
$$\mu_j \notin \left(\bigcup_{i=1}^{\infty} \mathrm{Spec}(C_i)\right) \cup \mathrm{Spec}(A).$$

Put $B_i := P \begin{pmatrix} \mu_i & b_{12} \dots b_{1n} \\ 0 & \\ \vdots & C_i \\ 0 & \end{pmatrix} P^{-1}$

Then 1) B_i is diagonalizable $/\mathbb{C}$ (Exercise)

2) $B_i \xrightarrow{i \to \infty} A$ w.r.t. $\|\cdot\|_{\mathrm{Eu},n}$.

$\square$

Example 2.46: $A = \begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix}$

$$p_A(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$$\mathrm{Eig}_{\mathbb{C}}(A, 2) = \mathrm{null}\left(\begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}\right) = \mathbb{C}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$P := \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}, \quad P^{-1} A P = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 - \frac{1}{m} & -1 \\ & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 \\ & 2 \end{pmatrix}$$

$$\Rightarrow \underbrace{P \begin{pmatrix} 2 - \frac{1}{m} & -1 \\ & 2 \end{pmatrix} P^{-1}}_{\shortparallel} \longrightarrow A$$

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 - \frac{1}{m} & -1 \\ & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{m} - 4 & 4 \\ 2 - \frac{1}{m} & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & \frac{2}{m} + 4 \\ -1 & -\frac{1}{m} \end{pmatrix} = B_m.$$

Corollary 2.47 (Cayley-Hamilton Theorem)
Let $A \in \mathbb{C}^{n \times n}$. Then $p_A(A) = 0 \in \mathbb{C}^{n \times n}$!
(More precisely: $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ with $a_i \in \mathbb{C}$. We claim
$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \dots + a_1 A + a_0 I_n = 0$)

—2.45—

Proof: Case 1: If A is diagonalizable $/\mathbb{C}$

(Exercise!)

Case 2: If A is not diagonalizable $/\mathbb{C}$.

Prop. 2.45 $\Longrightarrow \exists\, B_j \xrightarrow{\|\cdot\|_{Eu}} A$, $B_j \in \mathcal{D}_n$.

$$\Longrightarrow \; P_A(A) = \lim_{j\to\infty} \underbrace{P_{B_j}(B_j)}_{0} = 0 \in \mathbb{C}^{n\times n}$$

$\square$

Question 2.48: Does the density statement also hold, if we replace $\mathbb{C}$ by $\mathbb{R}$ in ~~Thm~~ Prop. 2.45?

For $P \in \mathrm{Poly}(\mathbb{C})$ we denote

$\mathrm{zero}(P) := \{ z \in \mathbb{C} \mid P(z) = 0 \}$

"the set of roots (or zeros) of P.

Proposition 2.49: (Euclidean algorithm)

Let $P_1, \dots, P_\ell \in \mathrm{Poly}(\mathbb{C})$ s.t.

$\underbrace{\mathrm{zero}(P_1) \cap \dots \cap z(P_\ell)}_{=: V(P_1, \dots, P_\ell)} = \emptyset$.

Then $\exists\, q_1, \dots, q_\ell \in \mathrm{Poly}(\mathbb{C})$: $\sum_{i=1}^{\ell} q_i P_i = 1$,

i.e. $\sum_{i=1}^{\ell} q_i(\lambda) P_i(\lambda) = 1 \;\forall \lambda \in \mathbb{C}$.

Proof: (Induction on ℓ)

Base case $\ell = 1$: $\operatorname{zero}(P_1) = \emptyset \underset{\substack{\uparrow \\ \text{FTA} \\ \text{(Thm 230)}}}{\Rightarrow} P_1 \in \mathbb{C} \setminus \{0\}$

Take $q_1 := \frac{1}{P_1} \in \mathbb{C}$.

$\Rightarrow \quad q_1(\lambda)P_1(\lambda) = 1 \quad \forall \lambda \in \mathbb{C}$.

Induction step $\ell > 1$: We prove

(*) $\exists\, a, b \in \operatorname{Poly}(\mathbb{C}) : \operatorname{zero}(a P_1 + b P_2) = V(P_1, P_2)$.

Then apply the induction hypothesis (case $\ell - 1$) to $aP_1 + bP_2, P_3, \dots, P_\ell$.

Proof of (*):

$$P_1 = P_2 \cdot s_1 + r_1 \qquad \deg(r_1) < \deg(P_2) \text{ and } V(P_1, P_2) = V(P_2, r_1)$$

$$P_2 = r_1 s_2 + r_2 \qquad \deg(r_2) < \deg(r_1) \text{ and } V(P_2, r_1) = V(r_2, r_1)$$

$$r_1 = r_2 s_3 + r_3 \qquad \deg(r_3) < \deg(r_2) \quad V(r_1, r_2) = V(r_2, r_3)$$

$$\vdots$$

$$r_{k-2} = r_{k-1} s_k + r_k \qquad \deg(r_k) < \deg(r_{k-1}) \quad V(r_{k-2}, r_{k-1}) = V(r_{k-1}, r_k)$$

$$r_{k-1} = r_k s_{k+1} + 0 \qquad V(r_{k-1}, r_k) = \underset{\substack{\parallel \\ \operatorname{zero}(r_k)}}{V(r_k)}.$$

So $r_k = r_{k-2} - r_{k-1} s_k$

Backwards substitution $\Rightarrow \exists\, a, b \in \operatorname{Poly}(\mathbb{C})$:

$r_k = a P_1 + b P_2$ and

$V(r_k) = V(P_1, P_2)$ $\square$

Example: $P_1(\lambda) = (\lambda - 1)\lambda = \lambda^2 - \lambda$

$P_2(\lambda) = (\lambda - 1)(\lambda + 2) = \lambda^2 + \lambda - 2$

$$P_1(\lambda) = 1 \cdot P_2(\lambda) + \underbrace{(2 - 2\lambda)}_{r_1}$$

$$P_2(\lambda) = (\lambda + 2) \cdot \frac{(-1)}{2}(2 - 2\lambda) + 0$$

$$\Rightarrow \quad 2 - 2\lambda = 1 \cdot P_1(\lambda) - 1 \cdot P_2(\lambda)$$

$$\Rightarrow \quad (\lambda - 1) = \left(\frac{-1}{2}\right) P_1(\lambda) + \frac{1}{2} P_2(\lambda).$$

<u>Theorem 250</u>: Let $A \in \mathbb{C}^{n \times n}$ with $P_A(\lambda) = (\lambda - \lambda_1)^{\nu_1} \cdots (\lambda - \lambda_\ell)^{\nu_\ell}$.

$\lambda_1, \ldots, \lambda_\ell$ distinct.

Then (1) $\mathbb{C}^n = \operatorname{Eig}_{\mathbb{C}}^{\nu_1}(A, \lambda_1) + \ldots + \operatorname{Eig}_{\mathbb{C}}^{\nu_\ell}(A, \lambda_\ell)$ (direct)

(2) $\operatorname{Eig}_{\mathbb{C}}^{\nu_i}(A, \lambda_i)$ is A invariant, i.e.

$$A v \in \operatorname{Eig}_{\mathbb{C}}^{\nu_i}(A, \lambda_i) \quad \forall v \in \operatorname{Eig}_{\mathbb{C}}^{\nu_i}(A, \lambda_i)$$

<u>Proof</u>: (1) $P_i(\lambda) := (\lambda - \lambda_1)^{\nu_1} \cdots (\lambda - \lambda_{i-1})^{\nu_{i-1}} (\lambda - \lambda_{i+1})^{\nu_{i+1}} \cdots (\lambda - \lambda_\ell)^{\nu_\ell}$.

(We omit $(\lambda - \lambda_i)^{\nu_i}$).

Then $V(P_1, \ldots, P_\ell) = \emptyset$

$\underset{\text{Prop. 249}}{\Longrightarrow}$ $\exists\, q_1, \ldots, q_\ell \in \operatorname{Poly}(\mathbb{C})$: $q_1 P_1 + \ldots + q_\ell P_\ell = 1$.

$\Rightarrow$ $q_1(A) P_1(A) + \ldots + q_\ell(A) P_\ell(A) = I_n$ (**)

(This is true for diagonalizable matrices. Use density Prop. 245 to get the eqn for A.)

$v \in \mathbb{C}^n$. $v_i := q_i(A)\, p_i(A) v$

$\Rightarrow$ $v = I_n v = v_1 + \dots + v_\ell$ (**)

and $(A - \lambda_i I_n)^{\nu_i} v_i = P_A(A)\, q_i(A) v$

$$\underset{\uparrow}{=} 0\, q_i(A) v = 0_{\mathbb{C}^n}$$

Cayley-Hamilton
Corollary 247

Sum (1) is direct: Exercise

(2) $(A - \lambda_i I)^{\nu_i} v = 0 \Rightarrow (A - \lambda_i I)^{\nu_i} A v$ A and I commute

$$\parallel$$

$$A (A - \lambda_i I)^{\nu_i} v = 0 \qquad \square$$

<u>Proof of Theorem 235</u> $B_i \subseteq \mathrm{Eig}^{\nu_i}_{\mathbb{C}}(A, \lambda_i)$ a basis. $B = B_1 \cup \dots \cup B_\ell$ a basis of $\mathbb{C}^n$.

$$\Rightarrow [T_A]_{B} = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_\ell \end{pmatrix} \qquad A_i \in \mathbb{C}^{d_i \times d_i}$$

$$d_i := \dim_{\mathbb{C}} \mathrm{Eig}^{\nu_i}_{\mathbb{C}}(A, \lambda_i)$$

$P_A(\lambda) = P_{A_1}(\lambda) \cdots P_{A_\ell}(\lambda)$.

<u>Claim</u>: $P_{A_i}(\lambda) = (\lambda - \lambda_i)^{d_i}$.

<u>Pf</u>: Let $\lambda_0 \in \mathrm{Spec}(A_i)$, $v \in \mathbb{C}^{d_i} \setminus \{0\}$

$A_i v = \lambda_0 v$.

$\Rightarrow 0 = (A_i - \lambda_i I_{d_i})^{\nu_i} v = (\lambda_0 - \lambda_i)^{\nu_i} v$

$\underset{v \neq 0}{\Rightarrow}$ $\lambda_i = \lambda_0$. Thus $\mathrm{Spec}(A_i) = \{\lambda_i\}$ $\square$

So $P_A(\lambda) = (\lambda - \lambda_1)^{d_1} \cdots (\lambda - \lambda_\ell)^{d_\ell}$ and $= (\lambda - \lambda_1)^{\nu_1} \cdots (\lambda - \lambda_\ell)^{\nu_\ell}$

$\Rightarrow d_i = \nu_i \ \forall i = 1, \dots, \ell$. $\square$