Chapter V

Eigenvalues and Eigenvectors

Def 207 1) Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. We call λ an ~~Eigenva~~ eigenvalue of A (of T_A) if $\exists v \in \mathbb{R}^n \setminus \{0\}$: $Av = \lambda v$.
Every such vector is called eigenvector of A corresponding to λ

2) Let V be a vector space and $f: V \to V$ be a linear map and $\lambda \in \mathbb{R}$.
We call λ an eigenvalue of f if $\exists v \in V \setminus \{0_V\}$: $f(v) = \lambda v$.
Every such vector is called eigenvector of f corresponding to f.

3) A ~~vector is~~ non-zero vector is called eigenvector of A (of T_A, of f)

if it is an eigenvector corresponding to some eigenvalue of (A, Γ_A, f).

<u>Notation 208</u>: (a) $\mathrm{Spec}_{\mathbb{R}}(A) = \{\lambda \in \mathbb{R} \mid \lambda$ eigenvalue of $A\}$
"real spectrum of A"

$\mathrm{Spec}_{\mathbb{R}}(f) = \{\lambda \in \mathbb{R} \mid \lambda$ eigenvalue of $f\}$
"real spectrum of f"

(b) $\mathrm{Eig}(A, \lambda) := \{v \in \mathbb{R}^n \mid Av = \lambda v\}$
"eigenspace of A corresponding to λ"
$\mathrm{Eig}(f, \lambda) = \{v \in V \mid f(v) = \lambda v\}$
"eigenspace of f corresponding to λ".

<u>Important remark 209</u>: An eigenvector ~~always~~ always has to be non-zero. So $\mathrm{Eig}(A, \lambda)$ does contain a vector which is not an eigenvector, namely $\underline{0}$.

Examples 210:

(a) $A = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$

Solve $Av = \lambda v$ for $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^{2\times 1} \setminus \{0\}$

Sol^n:

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} v = \lambda v \iff \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ +v_2 \end{pmatrix}$$

$\iff$ Case $v_2 \neq 0$: $\lambda = -1$ and $v \in \mathbb{R}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Case $v_2 = 0$ and $v_1 \neq 0$:

$\lambda = 1$ and $v \in \mathbb{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

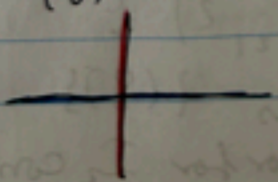
So $\mathrm{Spec}_{\mathbb{R}}(A) = \{1, -1\}$

$\mathrm{Eig}(A, 1) = \mathbb{R} e_1$

$\mathrm{Eig}(A, -1) = \mathbb{R} e_2$.

Geometry: T_A preserves the lines

$L_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ and $L_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$



, but no other lines.

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(b) $A = \begin{pmatrix} 1 & 1 \\ & 2 \end{pmatrix}$

$Av = \lambda v \quad , v \neq \underline{0}$

$\Leftrightarrow \begin{pmatrix} v_1 + v_2 \\ 2v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Case 1 $v_2 \neq 0$: $\lambda = 2$ and $v_1 = v_2$

$\mathrm{Eig}(A, 2) = \mathbb{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Case 2 $v_2 = 0$: $\lambda = 1$ and $v_1 \in \mathbb{R} \setminus \{0\}$

$\mathrm{Eig}(A, 1) = \mathbb{R}\binom{1}{0}$.

$\mathrm{Spec}_{\mathbb{R}}(A) = \{1, 2\}$

(c) $A = I_2$

$Av = \lambda v \quad , v \neq \underline{0}$

$I_2 v = \lambda v \Leftrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\underset{v \neq \underline{0}}{\Leftrightarrow} \quad \lambda = 1 \quad , v \in \mathbb{R}^{2\times 1} \setminus \{\underline{0}\}$

$\mathrm{Eig}(I_2, 1) = \mathbb{R}^{2\times 1}$

$\mathrm{Spec}_{\mathbb{R}}(I_2) = \{1\}$

What is $\mathrm{Eig}(I_2, 2)$?

Answer: It is $\{\binom{0}{0}\}$

There is no eigenvector for I_2 corresp. to 2.

We have two tasks:

1. Find the eigenvalues of A
2. Find the eigenvectors corresponding to an eigenvalue.

For Task ①: Use characteristic polynomial

Task ②: Use linear system

Def 211: (a) Let $A \in \mathbb{R}^{n \times n}$. The polynomial $p_A \in \mathrm{Poly}(\mathbb{R})$

defined by

$$p_A(\lambda) := \det(I_n \lambda - A)$$

is called <u>characteistic polynomial of A</u>

(b) Let V be a f.d. vector space and $f : V \to V$ be a linear map. Take a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V.

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Put $A := [f]_B$.

$p_f := p_A$ is called the

<u>characteristic polynomial</u> of f.

<u>Remark 2.12:</u> Why is p_f independent of the choice of basis?

<u>Examples 2.13:</u> 1) $p_{I_2}(\lambda) = \det(\lambda I_2 - I_2)$

$$= |(\lambda - 1) I_2| = (\lambda - 1)^2.$$

2) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$p_A(\lambda) = \det(\lambda I_2 - A)$$

$$= (-1)^2 \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$= (+1) \cdot [(a-\lambda)(d-\lambda) - cb]$$

$$= \lambda^2 - (a+d)\lambda + (ad - cb)$$

$$= \lambda^2 - \operatorname{tr}(A)\lambda + \det(A).$$

3) $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ & 1 & 2 \end{pmatrix}$

$$p_A(\lambda) = (-1)^3 \left| A - \lambda I_3 \right|$$

$$= - \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{vmatrix}$$

$$= (-1)(2-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (\lambda - 2)\left[(1-\lambda)^2 + 1\right]$$

$$= (\lambda - 2)\left[\lambda^2 - 2\lambda + 2\right]$$

$$= \lambda^3 - 2\lambda^2 + 2\lambda$$

$$\cancel{+\lambda^2} - 2\lambda^2 + 4\lambda - 4$$

$$= \lambda^3 - 4\lambda^2 + 6\lambda - 4$$

Theorem 214: Take $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. Then are equivalent

1° λ is an eigenvalue of A

2° $p_A(\lambda) = 0$ (We say λ is a root of p_A)

Proof: $1^\circ \Rightarrow 2^\circ$ Suppose $\lambda \in \mathrm{Spec}_{\mathbb{R}}(A)$.

Then $\exists\, v \in \mathbb{R}^n \setminus \{\underline{0}\} : Av = \lambda v = \lambda I_n v$

$\Rightarrow \exists_{v \neq \underline{0}} : (\lambda I_n - A)v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$\underset{\substack{\uparrow \\ v \neq \underline{0}}}{\Rightarrow}$ $(\lambda I_n - A)$ is not invertible

$\overset{\text{Thm. 82}}{\Rightarrow}$ $\det(\lambda I_n - A) = 0_{\mathbb{R}}$

$2^\circ \Rightarrow 1^\circ$ The above implications can be reversed. $\square$

Examples 215: (a)

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

$$p_A(\lambda) = (-1)^3 \begin{vmatrix} 1-\lambda & 1 & -1 \\ 2 & 1-\lambda & 1 \\ 2 & 2 & 1-\lambda \end{vmatrix}$$

$$= - \begin{vmatrix} 0 & 0 & -1 \\ -\lambda+3 & 2-\lambda & 1 \\ 2+(\lambda-1)^2 & 3-\lambda & 1-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 3-\lambda & 2-\lambda \\ \lambda^2-2\lambda+3 & 3-\lambda \end{vmatrix} = \lambda^2-6\lambda+9 + (\lambda-2)(\lambda^2-2\lambda+3)$$

$$= -\lambda^2-2\lambda+3 + \lambda^3-2\lambda^2+3\lambda = \lambda^3-3\lambda^2+\lambda+3$$

So $p_A(\lambda) = \lambda^3-3\lambda^2+\lambda+3$

$$= (\lambda-1)^3+(-2)\lambda+4.$$

$$\lambda_1 = 1 - \frac{2}{\sqrt[3]{27-3\sqrt{57}}} - \frac{\sqrt[3]{9-\sqrt{57}}}{3^{2/3}}$$

λ_2, λ_3 two roots in $\mathbb{C}\setminus\mathbb{R}$.

$\mathrm{Spec}_{\mathbb{R}}(A) = \{\lambda_1\}$.

(e)

$$A = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 2 \end{pmatrix}$$

Eigenvalues: $p_A(\lambda) = \det(\lambda I_3 - A)$
$= (-1)^3 \det(A-\lambda I_3)$

$$= - \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)$$

$$\text{roots}(P_A) = \{1, 2\} = \mathrm{Spec}_{\mathbb{R}}(A)$$

Compute Eigenspaces:

$\mathrm{Eig}(A,1)$: $\quad Av = 1 \cdot v \Leftrightarrow (A - I_3)v = 0$

$$(A - I_3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathrm{Eig}(A,1) = \mathrm{null}(A - I_3)$$

$$= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\mathrm{Eig}(A,2)$: $\quad Av = 2v$

$$\mathrm{Eig}(A,2) = \mathrm{null}(A - 2I_3)$$

$$= \mathrm{null}\left(\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$= \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

Eigenvalues: $p_A(\lambda) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -2 & 5 & \lambda-4 \end{vmatrix}$

$$= \lambda^2(\lambda-4) - 2 + 5\lambda$$
$$= \lambda^3 - 4\lambda^2 + 5\lambda - 2$$
$$\underset{\substack{\uparrow \\ \lambda=1 \text{ is a} \\ \text{root}}}{=} (\lambda-1)(\lambda^2 - 3\lambda + 2) = (\lambda-1)^2(\lambda-2)$$

$\Rightarrow \text{Spec}_{\mathbb{R}}(A) = \{1, 2\}$

Eigenspaces:

$\underline{\lambda = 1}$: $\text{null}\left(\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}\right)$

$$= \text{null}\left(\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{pmatrix}\right) = \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 2}$: $\operatorname{Eig}(A, 2) = \operatorname{null}\left(\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix}\right)$

$$= \operatorname{null}\left(\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{pmatrix}\right) = \mathbb{R}\begin{pmatrix} \frac{1}{2} \\ 1 \\ 2 \end{pmatrix}$$

<u>Prop 2.16:</u> Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be distinct eigenvalues of $A \in \mathbb{R}^{n \times n}$ and $v_1, \dots, v_r \in \mathbb{R}^{n \times 1}$ s.t. v_i is an eigenvector of A corresp. to λ_i.
Then $\{v_1, \dots, v_r\}$ is linearly independent.

<u>Proof:</u> $\sum_{i=1}^{r} \mu_i v_i = 0$.

$\Rightarrow$ $0 = A\left(\sum \mu_i v_i\right) = \sum \mu_i (A v_i) = \sum \mu_i \lambda_i v_i$

apply A

$\Rightarrow$ $0 = \sum \mu_i \lambda_i^2 v_i$

$\vdots$

$\Rightarrow$ $0 = \sum \mu_i \lambda_i^{r-1} v_i$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & & & \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \cdots & \lambda_r^{r-1} \end{pmatrix}}_{B} \begin{pmatrix} \mu_1 \vec{v}_1 \\ \mu_2 \vec{v}_2 \\ \vdots \\ \mu_r \vec{v}_r \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \\ \vdots \\ \vec{0} \end{pmatrix}$$

$\Rightarrow$ multiply with $Adj(B)$ from the left

$$\begin{pmatrix} \vec{0} \\ \vdots \\ \vec{0} \end{pmatrix} = Adj(B)\, B \begin{pmatrix} \mu_1 \vec{v} \\ \vdots \\ \mu_r \vec{v}_r \end{pmatrix}$$

$$= \det(B)\, I_r \begin{pmatrix} \mu_1 \vec{v}_1 \\ \vdots \\ \mu_r \vec{v}_r \end{pmatrix}$$

$$= \begin{pmatrix} \det(B)\mu_1 \vec{v}_1 \\ \vdots \\ \det(B)\mu_r \vec{v}_r \end{pmatrix}$$

This trick is called Noether's determinant trick.

Now $\vec{v}_i \neq \vec{0}$ because it is an eigenvector. So $\det(B)\mu_i = 0$ (Why?)

$$\det(B) \underset{\substack{\uparrow \\ \text{exercise}}}{=} \prod_{i>j} (\lambda_i - \lambda_j) \neq 0.$$

Thus all μ_i are zero $\square$

Corollary 217: (Exercise)
Suppose $\lambda_1, \ldots, \lambda_r$ are distinct eigenvalues of A and B_i is a basis of $Eig(A, \lambda_i)$.
Then $B = B_1 \cup B_2 \cup \ldots \cup B_r$ is linearly independent.

Rk 218: The set B in Cor 217 is a set of eigenvectors.
What happens if B is a basis of $\mathbb{R}^n$?

Example 2.19:

(a) $A = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

$\mathrm{Spec}_{\mathbb{R}}(A) = \{-1, 1, 2\}$

$\mathrm{Eig}(A, 1) = \mathbb{R}\vec{e}_1$

$\mathrm{Eig}(A, -1) = \mathbb{R}\vec{e}_2$

$\mathrm{Eig}(A, 2) = \mathbb{R}\vec{e}_3$

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is a basis of $\mathbb{R}^3$.

(b) $A = \begin{pmatrix} 1 & 1 & \\ & -1 & 1 \\ & & 2 \end{pmatrix}$

$P_A(\lambda) = (\lambda - 1)(\lambda + 1)(\lambda - 2)$

$\Rightarrow \mathrm{Spec}_{\mathbb{R}}(A) = \{1, -1, 2\}$

We can see that $\mathbb{R}^3$ has a basis consisting of eigenvectors of A by Prop. 2.16.

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Let's compute it

$$\mathrm{Eig}(A,1) = \mathrm{null}\left(\begin{pmatrix} 0 & 1 & \\ & -2 & 1 \\ & & 1 \end{pmatrix}\right) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{Eig}(A,-1) = \mathrm{null}\left(\begin{pmatrix} 2 & 1 & \\ & 0 & 1 \\ & & 3 \end{pmatrix}\right) = \mathbb{R}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\mathrm{Eig}(A,2) = \mathrm{null}\left(\begin{pmatrix} -1 & 1 & \\ & -3 & 1 \\ & & 0 \end{pmatrix}\right) = \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

<u>Check</u>: $\begin{pmatrix} 1 & 1 & \\ & -1 & 1 \\ & & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = (-1)\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Prop. 216 $\Rightarrow \mathcal{B} = \left\{ \overset{\vec{v}_1}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \overset{\vec{v}_2}{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}, \overset{\vec{v}_3}{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}} \right\}$ is linearly independent $\underset{\text{cardinality } 3}{\Rightarrow}$ basis of $\mathbb{R}^3$.

Put $P := \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

Columns are eigenvectors of A

$\Rightarrow \quad AP = P\begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

So $\quad P^{-1}AP = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

Interpretation with coordinate change:

$$T_A : \mathbb{R}^3 \to \mathbb{R}^3 \quad v \mapsto Av.$$

$$[T_A]_{\mathcal{B}} = \left([T_A(v_1)]_{\mathcal{B}}, [T_A(v_2)]_{\mathcal{B}}, [T_A(v_3)]_{\mathcal{B}} \right)$$

$$= \left([\vec{v}_1]_{\mathcal{B}}, [-\vec{v}_2]_{\mathcal{B}}, [2\vec{v}_3]_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and $[T_A]_{\mathcal{B}} = P_{\mathcal{A} \to \mathcal{B}} [T_A]_{\mathcal{A}} \underbrace{\left(P_{\mathcal{A} \to \mathcal{B}}\right)^{-1}}_{P_{\mathcal{B} \to \mathcal{A}}}$

$$P_{\mathcal{B} \to \mathcal{A}} = \left([v_1]_{\mathcal{A}}, [v_2]_{\mathcal{A}}, [v_3]_{\mathcal{A}} \right)$$

$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} = P.$$

Def 2.20: (a) $A, C \in \mathbb{R}^{n\times n}$ are called similar if $\exists\, P \in \mathbb{R}^{n\times n}$ invertible s.t. $P^{-1}AP = C$. Write $A \sim C$.

(b) $A \in \mathbb{R}^{n\times n}$ is called diagonalizable if A is similar to a diagonal matrix.

Rk 2.21: Similar matrices share the trace, the determinant and the characteristic polynomial.

Q: What about eigenvalues?
— " — eigenvectors?

Example 2.22: (i) $\begin{pmatrix} 1 & 1 & \\ & -1 & 1 \\ & & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

(ii) Similarity operations.

$(E3)_{sim}$ $\quad E_{ij}(\lambda)\, A\, \underbrace{E_{ij}(-\lambda)}_{j^{th} \text{ col} - \lambda\, i^{th} \text{ col.}}$

<u>Ex:</u> $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$(E2)_{sim}$ $\quad E_{ij}\, A\, E_{ij}$

swapping i^{th} row with j^{th} row
and then i^{th} col with j^{th} col.

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$$

$(E1)_{sim}$ $\quad E_i(\lambda)\, A\, E_i(\lambda^{-1})$, $\lambda \in \mathbb{R}^*$

i^{th} row times λ
then i^{th} col times λ^{-1}.

$$\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2\lambda \\ 3\lambda^{-1} & 4 \end{pmatrix}$$

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<u>Example 222 (iii):</u>

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

row op   column op   P col.

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\sim$ I + IV row, IV − I col.

$$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & & & -1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\sim$ II + IV row, IV − II col

$$\begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 2 & -2 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$\sim$ III + IV row, IV − III col

$$\begin{pmatrix} 0 & 2 & 2 & -2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ -1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 2 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}$$

$\sim$ I − II row, II + I col

$$\begin{pmatrix} 0 & 0 & 2 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 2 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\sim$ I − III row, III + I col

$$\begin{pmatrix} 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\sim$ IV + $\frac{1}{2}$I row, I − $\frac{1}{2}$IV col

$$\begin{pmatrix} 0 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 & -4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad \begin{pmatrix} \frac{3}{2} & 1 & 1 & -1 \\ \frac{1}{2} & 1 & 0 & -1 \\ \frac{1}{2} & 0 & 1 & -1 \\ -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\substack{\mathrm{I}-\mathrm{IV}\ \text{row}\\ \mathrm{IV}+\mathrm{I}\ \text{col}}}{\sim} \begin{pmatrix} 2&0&0&-2\\ 0&2&0&0\\ 0&0&2&0\\ 0&0&0&-2 \end{pmatrix} \quad \overset{S}{\underbrace{\begin{pmatrix} 2&0&0&0\\ 0&2&0&0\\ 0&0&2&0\\ 0&0&0&-2 \end{pmatrix}}_{D}} \quad \underbrace{\begin{pmatrix} \frac{3}{2}&1&1&\frac{1}{2}\\ \frac{1}{2}&1&0&-\frac{1}{2}\\ \frac{1}{2}&0&1&-\frac{1}{2}\\ -\frac{1}{2}&0&0&\frac{1}{2} \end{pmatrix}}_{P}$$

Then $P^{-1}AP = D \quad (\Leftrightarrow \quad AP = PD)$

and $\mathrm{Eig}(A,2) = \mathbb{R}\begin{pmatrix}3\\1\\1\\-1\end{pmatrix} + \mathbb{R}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \mathbb{R}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$

$\mathrm{Eig}(A,-2) = \mathbb{R}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}.$

(iv) $A = \begin{pmatrix}1&1&1\\ &1&1\\ &&2\end{pmatrix}$

$$\begin{array}{cccc} \text{row} & \text{col} & P & \lambda \\ & \begin{pmatrix}1&1&1\\ &1&1\\ &&2\end{pmatrix} & \begin{pmatrix}1&&\\ &1&\\ &&1\end{pmatrix} & \end{array}$$

$$\underset{\substack{\mathrm{II}+\lambda\mathrm{III}\ \text{row}\\ \mathrm{III}-\lambda\mathrm{II}\ \text{col}\\ (\lambda=-1)}}{\sim} \begin{pmatrix}1&1&1\\ &1&-1\\ &&2\end{pmatrix} \begin{pmatrix}1&1&2\\ &1&0\\ &&2\end{pmatrix} \begin{pmatrix}1&&0\\ &1&1\\ &&1\end{pmatrix} \qquad \begin{aligned} 1+2\lambda-\lambda&=0\\ 1+\lambda&=0 \end{aligned}$$

$$\underset{\substack{\mathrm{I}+\lambda\mathrm{III}\ \text{row}\\ \mathrm{III}-\lambda\mathrm{I}\ \text{col}\\ (\lambda=-2)}}{\sim} \begin{pmatrix}1&1&-2\\ &1&0\\ &&2\end{pmatrix} \underbrace{\begin{pmatrix}1&1&0\\ &1&0\\ &&2\end{pmatrix}}_{T} \underbrace{\begin{pmatrix}1&0&2\\ &1&1\\ &&1\end{pmatrix}}_{P} \qquad \begin{aligned} 2+\lambda 2-\lambda&=0\\ \lambda&=-2 \end{aligned}$$

So $P^{-1}AP = T \underset{(\mathrm{I}\leftrightarrow\mathrm{II})}{\sim} T^T = P^T A^T (P^T)^{-1}$

So $A^T \sim A.$

In fact: $A^T = (P^{-1})^T E_{12} P^{-1} A P E_{12} P^T$

$P E_{12} P^T = \begin{pmatrix}1&0&2\\ &1&1\\ &&1\end{pmatrix}\begin{pmatrix}&1&\\ 1&&\\ &&1\end{pmatrix}\begin{pmatrix}1&0&0\\ 0&1&0\\ 2&1&1\end{pmatrix} = \begin{pmatrix}0&1&2\\ 1&0&1\\ 0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\ 0&1&0\\ 2&1&1\end{pmatrix} = \begin{pmatrix}2&3&2\\ 3&1&1\\ 2&1&1\end{pmatrix}$

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(~~iv~~V) A =

$$\begin{pmatrix} -8 & 33 & 38 & 173 & -30 \\ 0 & 0 & -1 & -4 & 0 \\ 0 & 0 & -5 & -25 & 1 \\ 0 & 0 & 1 & 5 & 0 \\ 4 & -16 & -19 & -86 & 15 \end{pmatrix}$$

$$\underset{(\mathrm{III}+5\mathrm{IV})}{\sim} \begin{pmatrix} -8 & 33 & 38 & -17 & -30 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 4 & -16 & -19 & 9 & 15 \end{pmatrix}$$

$$\underset{(\mathrm{II}+\mathrm{IV})}{\sim} \begin{pmatrix} -8 & 33 & 38 & -50 & -30 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 4 & -16 & -19 & 25 & 15 \end{pmatrix}$$

$$\underset{(\mathrm{I}+2\mathrm{V})}{\sim} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 4 & -16 & -19 & 25 & 7 \end{pmatrix}$$

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$$\Rightarrow P_A(\lambda) = (-1)^5 \begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 0 & 1 \\ 0 & 0 & 1 & -\lambda & 0 \\ 4 & -16 & -19 & 25 & 7-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} -\lambda & 1 & & \\ & -\lambda & & \\ & & -\lambda & \\ & & 1 & -\lambda \end{vmatrix} (\lambda - 7)$$

$$- \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & 1 & -\lambda \\ 4 & -16 & -19 & 25 \end{vmatrix}$$

$$= \lambda^5 - 7\lambda^4 - \Big(\overset{\lambda^2(25-19\lambda)}{\cancel{\lambda^2 16}} - 16\lambda + 4 \Big)$$

$$= \lambda^5 - 7\lambda^4 + 19\lambda^3 - 25\lambda^2 + 16\lambda - 4$$

$$\underset{\lambda=1}{=} (\lambda - 1)(\lambda^4 - 6\lambda^3 + 13\lambda^2 - 12\lambda + 4)$$

$$= (\lambda-1)^2 (\lambda^3 - 5\lambda^2 + 8\lambda - 4)$$

$$= (\lambda-1)^3 (\lambda^2 - 4\lambda + 4) = (\lambda-1)^3 (\lambda-2)^2$$

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A is not diagonalizable,
because
$\dim(\mathrm{Eig}(A,1)) = 1 < 3$

(vi) $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = A$

$P_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 1$

$\Rightarrow \mathrm{Spec}_{\mathbb{R}}(A) = \{1 \pm \sqrt{2}\}$

Thus A is diagonalizable.

$\mathrm{Eig}(A, 1+\sqrt{2}) = \mathrm{null}\left(\begin{pmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{pmatrix}\right)$

$= \mathbb{R}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

$\mathrm{Eig}(A, 1-\sqrt{2}) = \mathrm{null}\left(\begin{pmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{pmatrix}\right)$

$= \mathbb{R}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

So for $P = \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix}$

we get $P^{-1} A P = \underbrace{\begin{pmatrix} 1+\sqrt{2} & \\ & 1-\sqrt{2} \end{pmatrix}}_{D}$

What is A^{1000}?

<u>Answer</u>: $A^{1000} = \left(P D P^{-1}\right)^{1000}$

$$= P D^{1000} P^{-1}$$

$$= \begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (1+\sqrt{2})^{1000} & \\ & (1-\sqrt{2})^{1000} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2}\,\lambda_1^{1000} & -\sqrt{2}\,\lambda_2^{1000} \\ \lambda_1^{1000} & \lambda_2^{1000} \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(\lambda_1^{1000} + \lambda_2^{1000}) & \frac{1}{\sqrt{2}}(\lambda_1^{1000} - \lambda_2^{1000}) \\ \frac{1}{2\sqrt{2}}(\lambda_1^{1000} - \lambda_2^{1000}) & \frac{1}{2}(\lambda_1^{1000} + \lambda_2^{1000}) \end{pmatrix}$$

Let us summarize:

<u>Theorem 223:</u> Let $A \in \mathbb{R}^{n \times n}$.

Then are equivalent

1° A diagonalizable (over $\mathbb{R}$)

2° $\sum\limits_{\lambda \in \mathrm{Spec}_{\mathbb{R}}(A)} \dim \mathrm{Eig}(A, \lambda) = n$

3° $\mathbb{R}^n$ contains a basis only consisting of eigen-vectors of A.

<u>Proof:</u> 1° $\Leftrightarrow$ 3° See the Example 222.

2° $\Rightarrow$ 3° ✓ by Corollary 217

$1^\circ \Rightarrow 2^\circ$ If $\exists P \in \mathbb{R}^{n\times n}$ invertible s.t. $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Rightarrow \quad AP = P\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Rightarrow \quad \{c_1(P), \dots, c_n(P)\}$

is a set of eigenvectors of A.

It is a basis of $\mathbb{R}^n$, because $\mathrm{rk}(P) = n$.

$\Rightarrow \sum_{\lambda \in \mathrm{Spec}_{\mathbb{R}}(A)} \dim \mathrm{Eig}(A, \lambda) \geq n$ $\square$

Interlude: Complex vector spaces

Def 224: A triple $(V, +, \cdot_{\mathbb{C},V})$

with $+ : V\times V \to V$ and $\cdot_{\mathbb{C},V} : \mathbb{C}\times V \to V$ is called complex vector space if it satisfies the vector space axioms with $\mathbb{C}$ instead of $\mathbb{R}$.

Example 225: (i) $\mathbb{C}^n$ $\lambda\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} := \begin{pmatrix} \lambda z_1 \\ \vdots \\ \lambda z_n \end{pmatrix}$

for $\lambda \in \mathbb{C}$, $\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n$.

(ii) $Map([0,1], \mathbb{C})$

$$(\lambda f)(t) := \lambda(f(t))$$

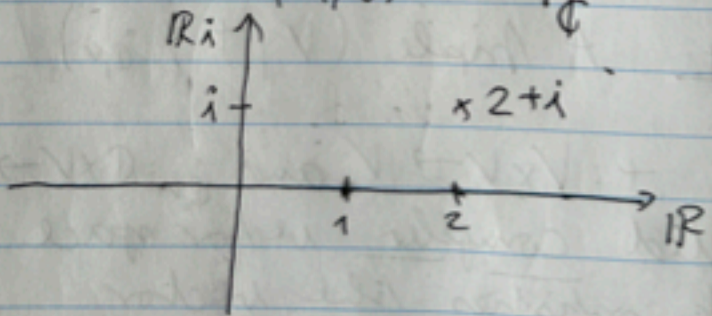
Recap 226: $\mathbb{C} = \mathbb{R}^2$, $\cdot_{\mathbb{C}}$ is given by $(a,b)(c,d) := (ac - bd, ad + bc)$.

$i := (0,1)$

$1_{\mathbb{C}} := (1,0)$

Then $(a,b) = (a,0) + (0,b)$
$= a(1,0) + b\,i$

Note $i^2 = (0,1) \cdot_{\mathbb{C}} (0,1) = (0^2 - 1^2, 0 \cdot 1 + 1 \cdot 0)$
$= (-1,0) = -1_{\mathbb{C}}$



We consider $\mathbb{R} \hookrightarrow \mathbb{C}$

$x \longmapsto (x,0)$

Exercise 227: Find an injective map $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}^{2\times 2}$ such that

$$\varphi(z_1 + z_2) = \varphi(z_1) + \varphi(z_2)$$
and
$$\varphi(z_1 z_2) = \varphi(z_1) \circ \varphi(z_2).$$

Def 228: We generalize for $A \in \mathbb{C}^{n\times n}$ the definition of eigenvalue and eigenvector in solving

$$Av = \lambda v \quad , \quad \lambda \in \mathbb{C}, \; v \in \mathbb{C}^n \setminus \{0\}$$

$$\mathrm{Spec}(A) := \{\text{eigenvalues of } A\}$$

$$= \{\text{roots of } p_A\}$$

Example 229: $A = \begin{pmatrix} 2 & 3 \\ -3 & -1 \end{pmatrix}$

$$p_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$$
$$= \lambda^2 - \lambda + 7$$

$$= \left(\lambda - \left(\frac{1}{2} + \frac{\sqrt{27}}{2} i\right)\right)\left(\lambda - \left(\frac{1}{2} - \frac{3\sqrt{3}}{2} i\right)\right)$$

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 7}$$

$$= (\lambda - \lambda_1)(\lambda - \overline{\lambda_1})$$

complex conjugation

$$\mathrm{Eig}_{\mathbb{C}}\left(A, \frac{1}{2} + \frac{3\sqrt{3}}{2} i\right)$$

$$= \mathrm{null}\left(\begin{pmatrix} \frac{3}{2}(1-\sqrt{3}i) & 3 \\ -3 & -\frac{3}{2}(1+\sqrt{3}i) \end{pmatrix}\right)$$

$$= \mathbb{C}\begin{pmatrix} -1 \\ \frac{1-\sqrt{3}i}{2} \end{pmatrix}$$

$$\mathrm{Eig}_{\mathbb{C}}\left(A, \frac{1}{2} - \frac{3\sqrt{3}}{2} i\right) = \mathbb{C}\begin{pmatrix} -1 \\ \frac{1+\sqrt{3}i}{2} \end{pmatrix}$$

<u>Check:</u> $\begin{pmatrix} 2 & 3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ \frac{1-\sqrt{3}i}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{3}\,3}{2} i \\ \frac{5}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix}$

$$= \left(\frac{1}{2} + \frac{3\sqrt{3}}{2} i\right) \left(\frac{-1}{\frac{1-\sqrt{3}i}{2}}\right) \checkmark$$

End of Lecture 13th Dec 23

Main property of $\mathbb{C}$:

Theorem 230 (Fundamental Theorem of Algebra)

Given a polynomial function $p \in \mathrm{Poly}(\mathbb{C})$ ~~xxx xxx~~ of degree ≥ 1. Then $\exists \lambda_0 \in \mathbb{C}: p(\lambda_0) = 0$.

Corollary 231: Let $p \in \mathrm{Poly}(\mathbb{C})$ be non-constant and monic. Then $\exists \lambda_1, \dots, \lambda_d \in \mathbb{C}$:

$$\forall_{\lambda \in \mathbb{C}} : p(\lambda) = (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_d)$$

Proof: p has a root λ_1 by Thm 230.

Euclidean algorithm

$\Rightarrow \quad p(\lambda) = (\lambda - \lambda_1) q(\lambda) + r$

with $r \in \mathbb{C}$.

$\underset{p(\lambda_1) = 0}{\Longrightarrow} \quad r = 0.$

$$= \left(\frac{1}{2} + \frac{3\sqrt{3}}{2} i\right)\left(\frac{-1}{\frac{1-\sqrt{3}i}{2}}\right) \checkmark$$

End of Lecture 13th Dec 23

Main property of $\mathbb{C}$:

Theorem 230 (Fundamental Theorem of Algebra)

Given a polynomial function $p \in \mathrm{Poly}(\mathbb{C})$ of degree ≥ 1. Then $\exists \lambda_0 \in \mathbb{C}: \; p(\lambda_0) = 0$.

Corollary 231: Let $p \in \mathrm{Poly}(\mathbb{C})$ be non-constant and monic. Then $\exists \lambda_1, \dots, \lambda_d \in \mathbb{C}$:
$\forall_{\lambda \in \mathbb{C}} : p(\lambda) = (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_d)$

Proof: p has a root λ_1 by Thm 230.

Euclidean algorithm
$\Rightarrow \quad p(\lambda) = (\lambda - \lambda_1)\, q(\lambda) + r$
with $r \in \mathbb{C}$.
$\underset{p(\lambda_1) = 0}{\Rightarrow} \quad r = 0.$

p is monic $\Rightarrow$ q is monic.
If q has degree ≥ 1
Then continue.

By induction
$$p(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_d). \quad \square$$

<u>Example 232</u>: (for Euclidean algorithm)

$$p(\lambda) = \lambda^4 + 5\lambda^3 + 3\lambda^2 + 2\lambda + 1$$
Find $q, r \in \mathrm{Poly}(\mathbb{R})$ such that
$$p(\lambda) = q(\lambda)(\lambda - 1) + \underset{\mathbb{R}}{\underset{\wedge}{r}}$$

$$\begin{aligned} p(\lambda) &= p(\lambda) - p(1) + p(1) \\ &= \lambda^4 + 5\lambda^3 + 3\lambda^2 + 2\lambda - 11 \; + 12 \\ &= (\lambda - 1)\underbrace{(\lambda^3 + 6\lambda^2 + 9\lambda + 11)}_{q(\lambda)} + \underbrace{12}_{r(\lambda)} \end{aligned}$$

We see that 1 is not a root of p.

So for $A \in \mathbb{R}^{n \times n}$ we have $\mathrm{Spec}(A) \neq \emptyset$.
It may happen that $\mathrm{Spec}_{\mathbb{R}}(A) = \emptyset$.

<u>Definition 233:</u> Let $A \in \mathbb{C}^{n \times n}$.
Then its characteristic polynomial has the form

$$p_A(\lambda) = (\lambda - \lambda_1)^{\nu_1} \cdot \ldots \cdot (\lambda - \lambda_r)^{\nu_r}$$

with distinct roots $\lambda_1, \ldots, \lambda_r$

Note: $\nu_1 + \ldots + \nu_r = n = \deg(p_A)$

We call

(i) ν_i the algebraic multiplicity of λ_i for A. We denote it by
$$m_a(A, \lambda_i)$$

(ii) $\dim_{\mathbb{C}} \mathrm{Eig}_{\mathbb{C}}(A, \lambda_i)$ the geometric multiplicity of λ_i for A.

We denote it by $m_g(A, \lambda_i)$.

Examples 234:

(a) $A = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix}$, $P_A(\lambda) = (\lambda-1)^2(\lambda-2)$

eig. value	1	2
m_a	2	1
m_g	2	1

$Eig(A, 1) = \mathbb{R}\begin{pmatrix}1\\0\\0\end{pmatrix} + \mathbb{R}\begin{pmatrix}0\\1\\0\end{pmatrix}$
has $\mathbb{R}$-dim 2

$Eig(A, 2) = \mathbb{R}\begin{pmatrix}0\\0\\1\end{pmatrix}$
has $\mathbb{R}$-dim 1

So here we have $m_a(A, \lambda) = m_g(A, \lambda)$
for $\lambda = \lambda_1, \lambda_c$

(b) $A = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 2 \end{pmatrix}$ $P_A(\lambda) = (\lambda-1)^2(\lambda-2)$

λ	1	2
m_a	2	1
m_g	1	1

$Eig(A, 1)$ is the subspace of $\mathbb{R}^3$ where A acts like multiplication with 1.

$$Eig^2(A, 1) = null\left((A - I_3)^{2 \leftarrow m_a}\right)$$

$$= null\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}\right) = \mathbb{R}\vec{e}_1 + \mathbb{R}\vec{e}_2$$

is the subspace where A acts like a shearing ~~[illegible]~~ along $Eig(A, 1)$

(c) $A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$$p_A(\lambda) = \lambda^2 - 0\lambda + 0 = \lambda^2$$

$$\Rightarrow Spec(A) = \{0\}$$

$$Eig(A, 0) = \mathbb{R}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$m_a(A, 0) = 2 > 1 = m_g(A, 0).$$

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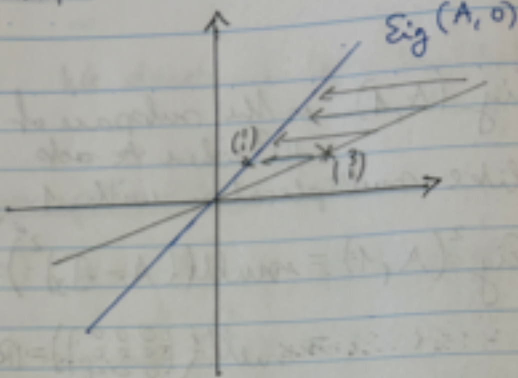

What does A do on the remaining part of $\mathbb{R}^2$?

$m_a(A, 0) = 2.$

So look at $\mathrm{Eig}^{m_a(A,0)}(A, 0)$

$= \mathrm{Eig}^2(A, 0) = \mathrm{null}\left((A - 0I_2)^2\right)$

$= \mathrm{null}(0) = \mathbb{R}^2.$

Given $v \in \mathbb{R}^2$: $Av \in \mathrm{Eig}(A, 0)$

Find v s.t. $Av = \binom{1}{1}$

Take $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\mathrm{Eig}^2(A, 0) = \mathbb{R}^2$

with the property $A \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

" ~~A cyclic basis~~ "A + cyclic basis".

<u>Theorem 235</u>: Let $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathrm{Spec}(A)$.
Then

$$m_a(A, \lambda) = \dim_{\mathbb{C}} \mathrm{Eig}_{\mathbb{C}}^{m_a(A,\lambda)}(A, \lambda)$$

$$= \dim_{\mathbb{C}} \mathrm{null}\left((A - \lambda I)^{m_a(A,\lambda)}\right)$$

<u>Proof</u>: later. $\square$

Corollary 236: $m_g(A, \lambda) \leq m_a(A, \lambda)$

<u>Proof</u>: $\mathrm{Eig}_{\mathbb{C}}(A, \lambda) \subseteq \mathrm{Eig}_{\mathbb{C}}^{m_a(A,\lambda)}(A, \lambda)$

$\Rightarrow$ $m_g(A, \lambda) \leq m_a(A, \lambda)$ $\square$

take $\dim_{\mathbb{C}}$

Example 236:

$$A = \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} \quad p_A(\lambda) = (\lambda - 2)^3$$

$$\mathrm{Eig}(A, 2) = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{Eig}^2(A, 2) = \mathrm{null}((A - 2I)^2)$$
$$= \mathrm{null}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbb{R}\vec{e}_1 + \mathbb{R}\vec{e}_2$$

$$\mathrm{Eig}^3(A, 2) = \mathrm{null}((A - 2I)^3)$$
$$= \mathrm{null}(0) = \mathbb{R}^3.$$

So we have

$$\underbrace{\mathrm{Eig}(A, 2)}_{\dim_{\mathbb{R}} = m_g} \subsetneq \mathrm{Eig}^2(A, 2) \subsetneq \underbrace{\mathrm{Eig}^3(A, 2)}_{\dim_{\mathbb{R}} = m_a}$$

Def 237: (i) A basis $B = \{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ is called ON basis if $v_i \perp v_j \ \forall i \neq j$ and $\|v_i\| = 1$ for all i.

(ii) A basis B of $\mathbb{R}^n$ is called eigenbasis for A if all $v \in B$ are eigenvectors of A.

Theorem 23.9 (symmetric matrices)

Let $A \in \mathbb{R}^{n\times n}$ be symmetric, i.e. $A = A^T$.

Then (1) $\mathrm{Spec}(A) = \mathrm{Spec}_{\mathbb{R}}(A)$, i.e. all eigenvalues of A are real.

(2) $\exists$ a basis of $\mathbb{R}^n$ consisting of eigenvectors of A ("eigenbasis")

This basis can be chosen such that it is an ON ~~basis~~ basis. (Gram - Schmidt later)

Proof: (1) Take $\lambda \in \mathrm{Spec}(A)$ and $v \in \mathrm{Eig}_{\mathbb{C}}(A, \lambda) \setminus \{\underline{0}\}$.

Then $\bar{v}^T A v = \bar{v}^T A v = \lambda \|v\|^2$

$$\begin{aligned} \bar{v}^T A v &= \bar{v}^T A v = \lambda \|v\|^2 \\ \| \\ (\bar{v}^T A) v &= \bar{\lambda}\, \bar{v}^T v = \bar{\lambda} \|v\|^2 \end{aligned}$$

because $\bar{v} \in \mathrm{Eig}_{\mathbb{C}}(\bar{A}^T, \bar{\lambda}) \underset{\uparrow}{=} \mathrm{Eig}_{\mathbb{C}}(A, \bar{\lambda})$

$A \in \mathbb{R}^{n\times n}$, A sym.

$\|v\| \neq 0 \Rightarrow \lambda = \bar{\lambda} \in \mathbb{R}$.

(2) $p_A(\lambda) = (\lambda - \lambda_1)^{\nu_1} \cdots (\lambda - \lambda_r)^{\nu_r}$

We have $\nu_1 + \ldots + \nu_r = \deg(p_A) = n$

By Thm 223 we need to show
$m_g(A, \lambda_i) \overset{(*)}{=} m_a(A, \lambda_i) = \nu_i$

Write $\lambda := \lambda_i$, $\nu := \nu_i$

If not $(*)$ then we have

By Thm 235

$Eig(A, \lambda) \subsetneq Eig^{\nu}(A, \lambda)$.

So $\exists\, j \geq 1$: $Eig^{j}(A, \lambda) \subsetneq Eig^{j+1}(A, \lambda)$

Apply $A - \lambda I$ $\quad j-1$ -times

$\Rightarrow \quad Eig(A, \lambda) \subsetneq Eig^{2}(A, \lambda)$.

Take $\quad v \in Eig^{2}(A, \lambda) \setminus Eig(A, \lambda)$

$\Rightarrow \quad 0_{\mathbb{R}} = v^{T} \underbrace{(A - \lambda I_n)^{2} v}_{\begin{pmatrix} 0 \\ \vdots \end{pmatrix}}$

$A = A^T$

$\Downarrow$

$= (v^T (A - \lambda I_n)^T)((A - \lambda I_n) v)$

$= \| (A - \lambda I_n) v \|^2$

$\Rightarrow v \in \mathrm{Eig}(A, \lambda)$? $\square$

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Example 239

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

$$p_A(\lambda) = (-1)^4 \begin{vmatrix} 1-\lambda & 1 & 1 & -1 \\ 1 & 1-\lambda & -1 & 1 \\ 1 & -1 & 1-\lambda & 1 \\ -1 & 1 & 1 & 1-\lambda \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 1 & 1 & 1-\lambda \\ 1-\lambda & 1 & 1 & -1 \\ 1 & 1-\lambda & -1 & 1 \\ 1 & -1 & 1-\lambda & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 1 & 1 & 1-\lambda \\ 0 & 2-\lambda & 2-\lambda & (1-\lambda)^2-1 \\ 0 & 2-\lambda & 0 & 2-\lambda \\ 0 & 0 & 2-\lambda & 2-\lambda \end{vmatrix}$$

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$$= \begin{vmatrix} 2-\lambda & 2-\lambda & (2-\lambda)(-\lambda) \\ 2-\lambda & 0 & 2-\lambda \\ 0 & 2-\lambda & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^3 \begin{vmatrix} 1 & 1 & -\lambda \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -(2-\lambda)^3 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -\lambda-1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (\lambda-2)^3 \left(1-(-\lambda-1)\right) = (\lambda-2)^3(\lambda+2$$

$$\mathrm{Eig}(A,2) = \mathrm{null}\left(\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}\right)$$

$$= \mathbb{R}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathrm{Eig}(A,-2) = \mathbb{R}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right.$$

is
an

is an eigen basis for A
and also an ON basis.