Chapter IV

General vector spaces.

Def 151: A triple $(V, +, \cdot)$ consisting of a set V and maps

$$+ : V \times V \longrightarrow V \quad \text{(addition)}$$
$$\cdot : \mathbb{R} \times V \longrightarrow V \quad \text{(scalar multiplication)}$$

is called real vector space if it satisfies the axioms of Prop 106, i.e.

(1a) $(\mathrm{ass}, +)$, $(0, +)$, $(\mathrm{inv}, +)$

(1b) $(\mathrm{com}, +)$

(2) $(\mathrm{ldist}, \cdot, +)$, $(\mathrm{rdist}, +_{\mathbb{R}}, \cdot)$
$(\mathrm{ass}, \cdot_{\mathbb{R}}, \cdot)$, $(\mathrm{unit}, \cdot)$

Remark 152: (1a) means that $(V, +_V)$ is a "group"
(1a) and (1b) means that $(V, +)$ is an "abelian group"

So a vector space is a triple $(V, +, \cdot)$ s.t.
$(V, +)$ is an abelian group which satisfies (2).

Example 153:

(a) $(\mathbb{R}, +, \cdot)$ is a real vector space because $(\mathbb{R}, +)$ is an abelian group and
(2) · (ldist) and (rdist) are satisfied, because $(+, \cdot)$ satisfies distributivity

· $(\text{ass}, \cdot_\mathbb{R}, \cdot_\mathbb{R})$ is satisfied as $\cdot_\mathbb{R}$ is associative

· $1_\mathbb{R} \cdot a = a \;\forall a \in \mathbb{R}$.

(b) Consider $\{0\} \subseteq \mathbb{R}$.
$(\{0\}, +, \cdot)$ is a vector space

as $0 + 0 = 0$ and $\lambda 0 = 0 \;\forall \lambda \in \mathbb{R}$

(c) $\mathbb{R}^{2\times 2}$. The space of 2 times 2 matrices is a vector space w.r.t.

$+ : \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$

$\cdot : \mathbb{R} \times \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$

$$A + B := \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$\lambda A := \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$

$(\mathbb{R}^{2\times 2}, +)$ is an abelian group

by

(ass, +): $(A+B)+C = \left((a_{ij}+b_{ij})+c_{ij}\right)_{ij}$

$$\underset{(\text{ass}, +_{\mathbb{R}})}{=} \left(a_{ij} + (b_{ij}+c_{ij})\right)_{ij} = A + (B+C)$$

(0, +): $0 + A = (0 + a_{ij})_{ij} \underset{(0, +_{\mathbb{R}})}{=} (a_{ij})_{ij}$

$$= (a_{ij}+0)_{ij} = A + 0$$

$$\left(0 = (0_{\mathbb{R}})_{ij} \right)$$

(add. inv. in $\mathbb{R}$)

(inv, +) For A take $B := (-a_{ij})_{ij}$

Then $A + B = (a_{ij} + (-a_{ij}))_{ij}$

$= 0 = ((-a_{ij}) + a_{ij})_{ij}$

(com, +) $A + B = (a_{ij} + b_{ij})$

$\underset{\text{(com, }+_{\mathbb{R}})}{\overset{\uparrow}{=}} (b_{ij} + a_{ij}) = B + A$

(2) is satisfied too:

· $\lambda(A + B) \overset{\text{Def }+}{=} (\lambda(a_{ij} + b_{ij}))$

$= (\lambda a_{ij} + \lambda b_{ij}) \overset{\text{Def }+}{=} (\lambda a_{ij}) + (\lambda b_{ij})$

$\underset{\text{Def.}}{=} \lambda A + \lambda B$

· $(\lambda_1 + \lambda_2) A \overset{\text{Def.}}{=} ((\lambda_1 + \lambda_2) a_{ij})$

$\underset{\text{(dist, }\mathbb{R})}{=} (\lambda_1 a_{ij} + \lambda_2 a_{ij}) \underset{\text{Def }+}{=} (\lambda_1 a_{ij}) + (\lambda_2 a_{ij})$

$\underset{\text{Def.}}{=} \lambda_1 A + \lambda_2 A$

· $(\lambda_1 \lambda_2) A \underset{\text{Def.}}{=} ((\lambda_1 \lambda_2) a_{ij}) \underset{(\text{ass}, \mathbb{R})}{=} (\lambda_1(\lambda_2 a_{ij}))$

$$\underset{\text{Def.}}{=} \lambda_1 (\lambda_2 a_{ij})_{ij} \underset{\text{Def.}}{=} \lambda_1 (\lambda_2 A)$$

- $1_{\mathbb{R}} A \underset{\text{Def.}}{=} (1_{\mathbb{R}} a_{ij})_{ij} \underset{(\text{unit}, \cdot_{\mathbb{R}})}{=} (a_{ij})_{ij} = A.$

(d) $Map(\mathbb{N}, \mathbb{R})$ = set of sequences with real terms

$$(f+g)(n) := f(n) + g(n)$$

$$(\lambda f)(n) := \lambda(f(n))$$

($Map(\mathbb{N}, \mathbb{R}), +)$ is an abelian group, because (comm, +) and (ass, +) follow from (comm, $+_{\mathbb{R}}$) and (ass, $+_{\mathbb{R}}$).

For the neutral element take $O : \mathbb{N} \to \mathbb{R}$, defined as $O(n) := 0_{\mathbb{R}}$

The additive inverse of f is $g : \mathbb{N} \to \mathbb{R}$, $g(n) := -f(n)$.

(2) The axioms (ldist), (rdist), $(\mathrm{ass}, {}_{\mathbb{R}}\cdot)$ and $(\mathrm{unit}, \cdot)$ are shown as for $\mathbb{R}^{2\times 2}$.

(e) A strange example

$$V := \mathbb{R}^{>0}$$

$$+_V : V \times V \longrightarrow V, \quad a +_V b := a \cdot_{\mathbb{R}} b$$

$$\cdot_V : \mathbb{R} \times V \longrightarrow V, \quad \lambda \cdot_V a := a^\lambda$$

$$(a, b \in V = \mathbb{R}^{>0}, \lambda \in \mathbb{R})$$

(1) $(V, +_V)$ is an abelian group (Exercise)

What is the neutral element?
" add. inverse of $v \in V$?

(2) (l dist)

$$\lambda \cdot_V (a +_V b) = \lambda \cdot_V (a \cdot_{\mathbb{R}} b) = (ab)^\lambda$$

$$\underset{\uparrow}{=} a^\lambda b^\lambda = (\lambda \cdot_V a) \cdot_{\mathbb{R}} (\lambda \cdot_V b)$$

(additivity of taking a power)

$= (\lambda \cdot_V a) +_V (\lambda \cdot_V b)$

(rdist), (ass, $\cdot_{\mathbb{R}}, \cdot_V$), (unit, $\cdot$) are exercises.

<u>Def 154:</u> (subspace)

Let $(V, +, \cdot)$ be an $\mathbb{R}$-vector space

We call a subset $W \subseteq V$ a <u>subspace</u> of V if

$(W, +|_{W \times W}, \cdot|_{\mathbb{R} \times W})$ is a vector space.

(in particular $+(W \times W) \subseteq W$ and $\cdot(\mathbb{R} \times W) \subseteq W$)

<u>Example 155:</u> (a) If $(V, +, \cdot)$ is a real vector space then $\{0_V\}$ is a subspace of V.

Pf: We only have to show

(i) $+(\{0_V\}\times\{0_V\})\subseteq\{0_V\}$

(ii) $\cdot(\mathbb{R}\times\{0_V\})\subseteq\{0_V\}$.

(iii) and 0_V is the add inverse of 0_V.

(i) $$0_V+0_V \underset{(0,+_V)}{=} 0_V$$

(ii) $$\lambda 0_V \underset{(0,+)}{=} \lambda(0_V+0_V) \underset{(\ell\text{ dist}}{=} \lambda 0_V + \lambda 0_V$$

$\overset{\Longrightarrow}{+(-(\lambda 0_V))}$ $$0_V = \lambda 0_V$$

(iii) By (i) we have $0_V+0_V=0_V$.

so $-0_V = 0_V$. $\square$

(b) $\mathbb{R}\times\{0\} = \{(r,0) \mid r\in\mathbb{R}\}$

is a subspace of $(\mathbb{R}^2, +, \cdot)$

$(+ : \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ \cdot : \mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2)$

because $\operatorname{im}\left(+\big|_{(\mathbb{R}\times\{0\})\times(\mathbb{R}\times\{0\})}\right)$

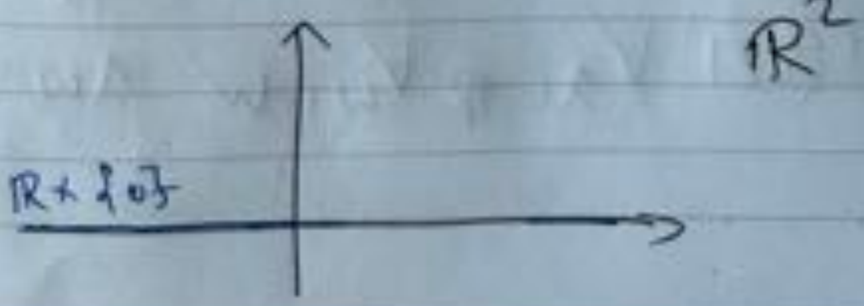
$$\subseteq \mathbb{R}\times\{0\}$$

$$\left((r_1,0) +_{\mathbb{R}^2} (r_2,0) = (r_1+r_2, 0+0) = (r_1+r_2,0) \in \mathbb{R}\times\{0\}\right)$$

and $\operatorname{im}\left(\cdot\big|_{\mathbb{R}\times(\mathbb{R}\times\{0\})}\right) \subseteq \mathbb{R}\times\{0\}$

$$\left(\lambda \cdot_{\mathbb{R}^2} (r_1, 0) = (\lambda r_1, 0) \in \mathbb{R}\times\{0\}\right)$$

and the next Lemma.



(c) $\mathbb{R} \times \{1\} = \{(r, 1) \mid r \in \mathbb{R}\}$ is not a subspace of $(\mathbb{R}^2, +, \cdot)$,

because

$$(1,1) + (0,1) = (1,2) \notin \mathbb{R} \times \{1\}.$$

(d) $\mathbb{Q} \subseteq (\mathbb{R}, +, \cdot)$ is not a (real) subspace ~~$\notin$~~, because

$$\sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Q}, \text{ i.e.}$$

$$\operatorname{im}\left(\cdot\big|_{\mathbb{R} \times \mathbb{Q}}\right) \not\subseteq \mathbb{Q}.$$

<u>Lemma 156:</u> Let $(V, +_V, \cdot_V)$ be a vector space. $\emptyset \neq W \subseteq V$ is a subspace of V iff

(S1) $\forall_{w_1, w_2 \in W}: w_1 +_V w_2 \in W$

(S2) $\forall_{\lambda \in \mathbb{R}} \forall_{w \in W}: \lambda w \in W$

In that case we have

$0_W = 0_V$ and

the additive inverse of w in $(V, +_V)$ is the " " " w in $(W, +_V|_{W \times W})$

<u>Proof:</u> Part 1: the equivalence.

"$\Rightarrow$": ✓ by definition of a subspace

"$\Leftarrow$" To show $(W, \overbrace{+|_{W \times W}}^{+_W}, \overbrace{\cdot|_{\mathbb{R} \times W}}^{\cdot_W})$ is a vector space.

$(W, +_W)$ is an abelian group:

(com, $+_W$) and (ass, $+_W$) are inherited from $+_V$.

$(0, +_W)$: Take $w_0 := 0_V$. Then for $w \in W$

$$w_0 + w = w + w_0 = w.$$

We have to show $w_0 \in W$.

$W \neq \emptyset \Rightarrow \exists\, w \in W$

(S2) $\Longrightarrow w_0 = 0_V \underset{\text{Exercise}}{\overset{\uparrow}{=}} 0_{\mathbb{R}} \cdot_V w \in W$

(inv, $+_W$) Take $w \in W$.

We have $\underbrace{(-1) \cdot_V w}_{=:\, \tilde{w}} \in W$

and $\tilde{w} +_V w \underset{(\text{unit}, \cdot)}{=} \tilde{w} +_V 1 \cdot_V w$

$= (-1) \cdot_V w + 1 \cdot_V w$

$\underset{(\text{r dist})}{=} ((-1) +_{\mathbb{R}} 1) \cdot_V w \underset{\substack{(-1) \text{ add inverse} \\ \text{of } 1 \text{ in } (\mathbb{R},+)}}{=} 0_{\mathbb{R}} \cdot_V w$

$\underset{\uparrow}{=} \; 0_V = w_0.$

Exercise

(2) (r dist), (l dist), (ass, $\mathbb{R}$, W), (unit, $\cdot$) are inherited from $(V, +_V, \cdot_V)$.

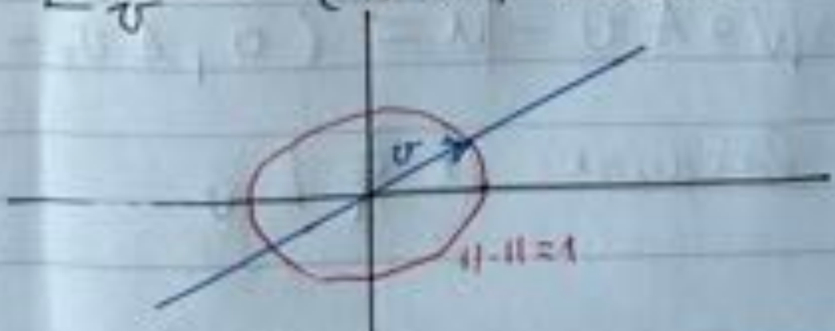
This also shows the second part of the Lemma.
(Note: neutral elt and add. inverse are unique.)

$\square$

<u>Example 157:</u> All subspaces of $(\mathbb{R}^2, +, \cdot)$:

- $\{\underline{0}\}$, $\mathbb{R}^2$, Else?
- Take $v \in \mathbb{R}^2$ with $\|v\| = 1$

$$L_v = \{ t v \mid t \in \mathbb{R} \}$$



Any more? No!

<u>Proof:</u> Let $W \subsetneq \mathbb{R}^2$ be a non-zero subspace.

Take $w \in W$ non-zero.
Then $v := \frac{1}{\|w\|} \cdot w \in W$ by (S2).

$\Rightarrow L_v \subseteq W$ by (S2)

To show $W = L_v$.

<u>Assume</u> $W \supsetneq L_v$.

Take $u \in W \setminus L_v$.

W.l.o.g $v_1 \neq 0$ $(v = (v_1, v_2))$

$\Rightarrow \exists_{A \in \mathbb{R}}: A v_1 = u_1$ $(u = (u_1, u_2))$

$\Rightarrow W \ni A v - u = (0, A v_2 - u_2) \neq 0$

because $u \notin L_v$

Thus $e_2 = (0,1) \in W_1$ by (52)

and $e_1 = (1,0) = \frac{1}{v_1}(v - v_2 e_2)$

$\in W$

Thus $\forall x = (x_1, x_2) \in \mathbb{R}^2$ we have

$x = x_1 e_1 + x_2 e_2 \in W$, i.e. $W = \mathbb{R}^2$ ✓

$\square$

End of Lecture 15th Nov 2023

Exercise 158: What about all subspaces in $\mathbb{R}^3$, or better all subspaces of $\mathbb{R}^3$ containing the line $L_{(1,1,0)}$?

Example 159: $W := (1,1) + L_{(1,0)} \subseteq \mathbb{R}^2$

is not a subspace. (Exercise.)

General constructions of subspaces

Prop 160: Let V be a vector space and $W_1, \dots, W_k$ be subspaces. Then

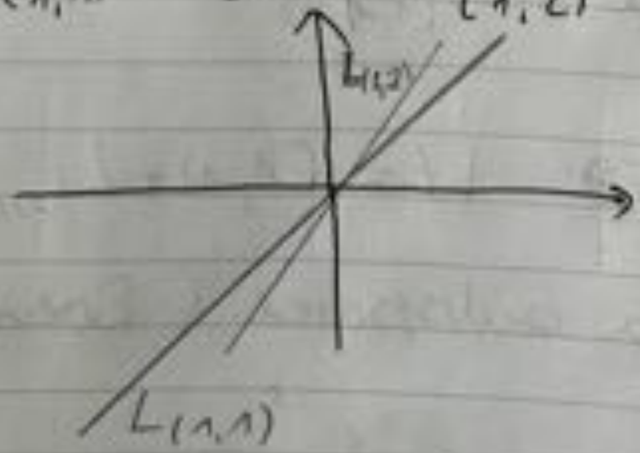
(a) The intersection $W_1 \cap \dots \cap W_k$ is a subspace of V

(b) The sum
$$W_1 + \dots + W_k := \{ w_1 + \dots + w_k \mid w_i \in W_i, \; i = 1, \dots, k \}$$
is a subspace of V.

Pf: Problem sheet 7. $\square$

Example 161:

(i) $L_{(1,1)}$ and $L_{(1,2)}$ in $\mathbb{R}^2$



$L_{(1,1)} \cap L_{(1,2)} = \{(0,0)\}$ zero space

$$L_{(1,1)} + L_{(1,2)} = \{ s(1,1) + t(1,2) \mid s,t \in \mathbb{R} \}$$
$$= \mathbb{R}^2$$

(for $(x_1,x_2) \in \mathbb{R}^2$, take $t = x_2 - x_1$ and $s = 2x_1 - x_2$, i.e.

$(x_1,x_2) = (2x_1 - x_2)\cdot(1,1) + (x_2 - x_1)(1,2)$.)

(ii) In $\mathbb{R}^4$. Consider the planes

$$E_1 = \{ s(1,0,0,0) + t(1,1,1,1) \mid s,t \in \mathbb{R} \}$$
$$E_2 = \{ s(1,0,0,0) + t(1,1,0,0) \mid s,t \in \mathbb{R} \}$$

$$E_1 \cap E_2 = \mathbb{R}(1,0,0,0)$$

Pf: "$\supseteq$" ✓ "$\subseteq$" Take $v \in E_1 \cap E_2$

Then $v = s(1,0,0,0) + t(1,1,1,1)$
$= \lambda(1,0,0,0) + \mu(1,1,0,0)$
for some $s, t, \lambda, \mu \in \mathbb{R}$.

$\Rightarrow (s - \lambda, 0,0,0) = (\mu - t, \mu - t, -t, -t)$
$\Rightarrow s = \lambda$ and $\mu = t = 0$.
$\Rightarrow v \in \mathbb{R}(1,0,0,0)$ $\square$

$$H_1 = E_1 + E_2 = \{ s(1,0,0,0) + t(1,1,1,1) + u(1,1,0,0) \mid s,t,u \in \mathbb{R} \}$$

We will see that this is a linear hypeplane (later).

Let's compute a normal

$\vec{e}_1 = (1,0,0,0)$, $\vec{e}_2 = (0,1,0,0)$, $\vec{e}_3$, $\vec{e}_4$

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$= 0 \cdot \vec{e}_1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \vec{e}_2$$

$$+ \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \vec{e}_3 - \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \vec{e}_4$$

$$= 0\,\vec{e}_1 - 0\,\vec{e}_2 + (-1)\,\vec{e}_3 - (-1)\,\vec{e}_4$$

$$= (0,0,-1,1)$$

So we get $H = H_{(0,0,-1,1)}$,

i.e. the linear hyperplane perpendicular to $(0,0,-1,1)$.

IV 2. Basis of a vector space

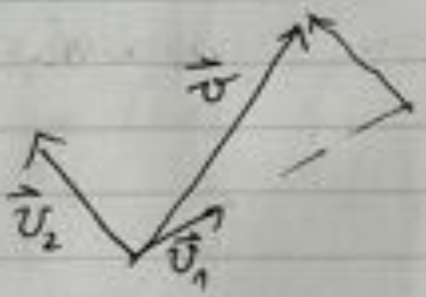
Let V be a vector space

Def 162: Take $\vec{v}, \vec{v}_1, \dots, \vec{v}_m \in V$.

We say that v is a linear combination of $v_1, \dots, v_m$ if $\exists\, \lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$\vec{v} = \lambda_1 \vec{v}_1 + \dots + \lambda_m \vec{v}_m$$

Picture:



$\vec{v} = 3\vec{v}_1 + \vec{v}_2$

Motivation for basis 163:

$$W = \mathbb{R}\begin{pmatrix}1\\2\\1\end{pmatrix} + \mathbb{R}\begin{pmatrix}1\\0\\2\end{pmatrix} + \mathbb{R}\begin{pmatrix}-1\\3\\-4.5\end{pmatrix} + \mathbb{R}\begin{pmatrix}5\\-9\\17.5\end{pmatrix}$$

$$= \left\{ \lambda_1\begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda_2\begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda_3\begin{pmatrix}-1\\3\\-4.5\end{pmatrix} + \lambda_4\begin{pmatrix}5\\-9\\17.5\end{pmatrix} \;\; \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \right\}$$

Part 1: Describe W with a small set of vectors.

Easy: Take $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -4.5 \end{pmatrix}, \begin{pmatrix} 5 \\ -9 \\ 17.5 \end{pmatrix} \right\}$

Part 2: We want to have the set as small as possible, i.e. no vector in the set is a linear combination of the other vectors.

Step 1: We see $\begin{pmatrix} 5 \\ -9 \\ 8.5 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ -4.5 \end{pmatrix}$

$\Rightarrow$ We can remove $\begin{pmatrix} 5 \\ -9 \\ 17.5 \end{pmatrix}$.

$S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -4.5 \end{pmatrix} \right\}$

Every vector $\in W$ is a lin. combination of vectors in S_1.

Step 2: Solve $\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 3 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Take $\lambda_1 = 1.5$, $\lambda_2 := -2.5$, $\lambda_3 := -1$

We can ~~rem~~ remove any of them.

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$$S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -4.5 \end{pmatrix} \right\}$$

Step 3: Solve

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 3 \\ -4.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \lambda_2 = 0 = \lambda_1$$

So we cannot make S_2 smaller.

Part 1 $\hat{=}$ Concept of generating set
Part 2 $\hat{=}$ Concept of linear independen

IV 2.1. Generating set

Prop 163: Let $S \subseteq V$. We put

$$U_1 := \bigcap_{\substack{W \supseteq S \\ W \text{ subspace of } V}} W \quad \text{and}$$

$$U_2 := \{ \lambda_1 v_1 + \ldots + \lambda_m v_m \mid m \in \mathbb{N}_0 \mid \lambda_1, \ldots, \lambda_m \in \mathbb{R} \text{ and } v_1, \ldots, v_m \in S \}$$

Then U_1 and U_2 are subspaces of V and $U_1 = U_2$.

Proof: Subspaces: Use subspace criterium (Lemma 156)

For U_1 exercise

For U_2: $U_2 \neq \emptyset$, because $0 \in U_2$ (0 is the empty sum)

$\lambda \in \mathbb{R}$, $\lambda_1 v_1 + \ldots + \lambda_m v_m \in U_2$

$\Rightarrow \lambda(\quad\quad) = (\lambda\lambda_1)v_1 + \ldots + (\lambda\lambda_m)v_m \in U_2$

$\lambda_1 v_1 + \ldots + \lambda_m v_m, \mu_1 w_1 + \ldots + \mu_n w_n \in U_2$

$\Rightarrow (\lambda_1 v_1 + \ldots + \lambda_n v_m) + (\mu_1 w_1 + \ldots + \mu_n w_n)$

$\in U_2$, because all $v_i, w_j \in S$ and $\lambda_i, \mu_j \in \mathbb{R}$.

Equality: "$\subseteq$" U_2 subspace of V, $U_2 \supseteq S$

$\Rightarrow U_1 \subseteq U_2$

"$\supseteq$" Take $\lambda_1 v_1 + \ldots + \lambda_m v_m \in U_2$

and $W \supseteq S$ a subspace. Then as $v_1, \ldots, v_m \in W$ we get $\lambda_1 v_1 + \ldots + \lambda_m v_m \in W$

So

$$- \text{"} - \quad \in \bigcap_{\substack{W \text{ subspace} \\ W \supseteq S}} W = U_1. \quad \square$$

Terminology 164:

$\mathrm{Span}(S) := \bigcap_{\substack{W \supseteq S \\ \text{subspace of } V}} W$ is called

the span of S

If $U = \mathrm{Span}(S)$ then we say that S spans U.

Example 165

(a) $H_{(0,0,1)} = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$

$= \mathbb{R}\underset{(1,0,0)}{\vec{e}_1} + \mathbb{R}\underset{(0,1,0)}{\vec{e}_2} = \mathrm{Span}(\{\vec{e}_1, \vec{e}_2\})$

(b) $H_{(1,1,1)} = \{\underline{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$

$\supseteq S := \{\underset{v_1}{(1,-1,0)}, \underset{v_2}{(0,1,-1)}, \underset{v_3}{(1,0,-1)}\}$

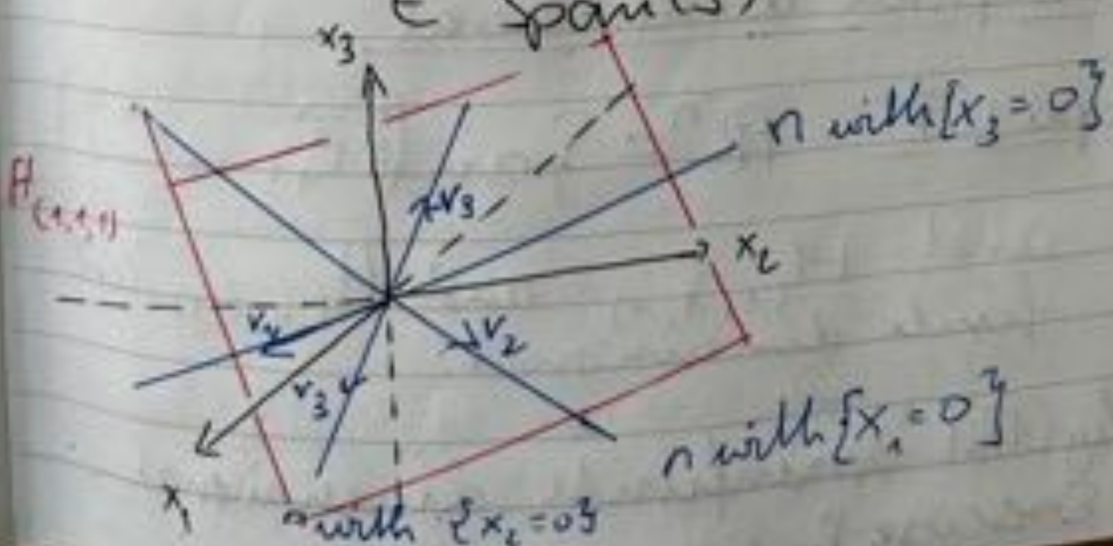
$\mathrm{Span}(S) = H_{(1,1,1)}$

Proof: "$\subseteq$" ✓

"$\supseteq$" Take $v = (x_1, x_2, x_3) \in H_{(1,1,1)}$

$\Rightarrow \quad v = x_1(1,-1,0) + (x_1 + x_2)(0,1,-1)$

$\in \mathrm{Span}(S)$ □



(c) $Poly(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f$ polynomial function $\}$

$\{$ a function of the form
$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$$
with $a_0, \dots, a_m \in \mathbb{R})$

Put $p_i(x) := x^i$
$\Rightarrow Poly(\mathbb{R}) = Span(\{p_0, p_1, p_2, \dots\})$

Def 166: V is called finite-dimensional (or finitely generated) if
$$\exists S \subseteq V \text{ finite} : Span(S) = V$$

otherwise V is called infinite dimensional

Example 167: $\mathbb{R}^n = Span(\{\vec{e}_1, \dots, \vec{e}_n\})$

and $H_{(1,1,1)}$ and $H_{(0,0,1)}$ are finite dimensional.

$Poly(\mathbb{R})$ is infinite dimensional.
(Exercise in example class)

IV 2.2. Linear independence

Def 168: A family of vectors $v_1, \dots, v_m \in V$ is called <u>linearly independent</u> if the equation

$$\lambda_1 v_1 + \dots + \lambda_m v_m = 0_V$$

has only the trivial solution, i.e. $\lambda_1 = \dots = \lambda_m = 0_{\mathbb{R}}$.

A finite set $S \subseteq V$, say $S = \{v_1, \dots, v_m\}$ with $v_i \neq v_j$ for $i \neq j$) is called <u>linearly independent</u> if the family $v_1, \dots, v_m$ is linearly independent.

Otherwise they are called <u>linearly dependent</u>

Rk 169: Linear indep for ∞ sets, see problem sheet.

Example 170: (a) in Motivation 163.
S is not linearly independent.
S_1 — "

S_2 is linearly independent.

(b) $\{0_V\}$ is linearly dependent.
as $1_{\mathbb{R}} \cdot 0_V = 0_V$.

(c) $v \in V \setminus \{0_V\}$.
$S = \{v\}$ is linearly independent.
Pf:
$\lambda v = 0_V$. Assume $\lambda \neq 0$
$\Rightarrow 0_V = \lambda^{-1}(\lambda v) = (\lambda^{-1}\lambda)\, v = 1_{\mathbb{R}} v = v$ ↯

(d) $\vec{e}_i = (0, \ldots, \overset{(i)}{1}, \ldots, 0) \in \mathbb{R}^n$
$S = \{\vec{e}_1, \ldots, \vec{e}_n\}$ is linearly indep.

Pf: $\sum_{i=1}^{n} \lambda_i \vec{e}_i = \underline{0}$
$\Rightarrow (\lambda_1, \lambda_2, \ldots, \lambda_n) = \underline{0}$
$\Rightarrow \lambda_1 = \ldots = \lambda_n = 0$ $\square$

(e) See 165(b)

$S = \{\underbrace{(1,-1,0)}_{v_1}, \underbrace{(0,1,-1)}_{v_2}, \underbrace{(1,0,-1)}_{v_3}\}$
is linearly dep. as

$v_1 + v_2 - v_3 = 0 \in \mathbb{R}^3$.

(f) $S = \{f_0, f_1, f_2\} \subseteq Map(\mathbb{R}, \mathbb{R})$

$f_0(x) = 1$
$f_1(x) = \cos x$
$f_2(x) = \cos(2x)$ $\quad , x \in \mathbb{R}$

Then S is lin. indep.

Pf: $\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 = 0$ - zero function.

Plug in: $x = 0$: $\lambda_0 + \lambda_1 + \lambda_2 = 0$
$x = \frac{\pi}{2}$: $\lambda_0 - \lambda_2 = 0$
$x = \frac{\pi}{4}$: $\lambda_0 + \frac{1}{\sqrt{2}} \lambda_1 = 0$
$x = \pi$: $\lambda_0 - \lambda_1 + \lambda_2 = 0$

$\Rightarrow \lambda_0 = \lambda_1 = \lambda_2 = 0$ $\square$

(g) $f_3 : \mathbb{R} \to \mathbb{R}$. $f_4(x) = \sin^2 x$

$S = \{f_0, f_1, f_2, f_3\}$ is lin. dependent

Pf: $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$ $\square$
$\Rightarrow 0 = f_0 - 2f_3 - f_2$.

IV 2.3. Basis

Lemma 171: Suppose $V = \mathrm{Span}(S)$ with $|S| < \infty$ and let $T \subseteq V$ be finite and linearly independent.
Then
$$|T| \leq |S|.$$

Proof: Assume $|T| > |S|$

$T = \{v_1, \dots, v_n\}$, $S = \{w_1, \dots, w_m\}$, $m < n$

Write $v_j = \sum_{i=1}^{m} a_{ij} w_i$, $a_{ij} \in \mathbb{R}$

Solve $\lambda_1 v_1 + \dots + \lambda_n v_n = 0_V$, e.g.
say $A \lambda^T = \underline{0}$, for
$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$.

Then
$$A \xrightarrow[\text{row operations}]{} R = \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & \vdots \end{matrix} \right.$$
r.r.e.f.

$m < n \Rightarrow \exists$ non-pivot column
$\Rightarrow \exists$ free variable λ_{i_0}

$\Rightarrow \exists$ solution $\lambda \in sol(A, \underline{0})$
with $\lambda_{i_0} = 1$
$\Rightarrow$ $\mathbf{v}_1, \dots, v_n$ are not lin. indep. ↯ □

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~~Def~~ Ex 172: $\nexists$ a lin. indep. set $T \subseteq \mathbb{R}^3$

with more than 3 elements, because
$\mathrm{Span}(\{e_1, e_2, e_3\}) = \mathbb{R}^3$.

Def 173: (basis) Let V be finite dim.
and $B \subseteq V$ finite.
We call B a basis of V if
B is linearly independent and
$\mathrm{Span}(B) = V$.

Rk 174: If V is infinite dimensional
$B \subseteq V$ is called a basis if
B is lin. indep. and spans V.

Example 175: (a) See 163: $S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -4,5 \end{pmatrix} \right\}$

is a basis of W.

(b) $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$

"standard basis of $\mathbb{R}^n$"

(c) $S = \{(1,-1,0),(0,1,-1)\}$
is a basis of $H_{(1,1,1)}$ (See 165 (b))
because S is generating $H_{(1,1,1)}$
and linearly indep.

$$\lambda_1(1,-1,0) + \lambda_2(0,1,-1) = 0$$

$\Rightarrow$ $\lambda_1 = 0$ and $\lambda_2 = 0$.

$\{(1,-1,0),(1,0,-1)\}$ is also a basis.

(d) $\{p_0, p_1, p_2, \dots\}$ is a basis

for $\mathrm{Poly}(\mathbb{R})$. (an example
for an infinite dimensional
vector space.)

<u>Theorem 176</u>: Let V be finite dimen-

sional and B_1, B_2 be bases for V.
Then $|B_1| = |B_2|$

Proof: B_1 generates V and B_2 is linearly independent $\Rightarrow |B_2| \leq |B_1|$ by Lemma 171.

$|B_2| \leq |B_1|$ similarly. $\square$

Def 177: Let V be finite dim. and $B \subseteq V$ a basis.

$\dim V := |B|$ is called the dimension of V.

Ex 178: $\dim \mathbb{R}^3 = 3$

$\dim H_{(1,1,1)} = 2$

In 163: $\dim W = 2$

$\dim \{0_V\} = 0$

$\dim L_v = 1$, $v \in V \setminus \{0\}$.
$(L_v = \mathbb{R}v)$

Example 179: The set

$$T = \left\{ \begin{array}{c} v_1 \\ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{array}, \begin{array}{c} v_2 \\ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \end{array}, \begin{array}{c} v_3 \\ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{array} \right\}$$ is linearly independent.

because

$$\mathrm{sol}\left(\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = \{\underline{0}\}$$

and $|T| = 3 = \dim \mathbb{R}^3$.

<u>Claim:</u> T spans $\mathbb{R}^3$

<u>Pf:</u> If not, take $v \in \mathbb{R}^3 \setminus \mathrm{Span}(T)$

Then $T \cup \{v\}$ is still linearly independent, because a non-zero sol^n of

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v = \underline{0}$$

must have $\lambda_4 \neq 0$ (why?)

Thus $v \in \mathrm{Span}(T)$ $\lightning$.

Lemma 171 $\Rightarrow |T \cup \{v\}| \leq 3$ $\lightning$

$\square$

More general:

<u>Theorem 180</u>: Let V be f.d. and $B \subseteq V$ be a finite subset. T.a.e.:

1° B is a basis of V
2° B is a minimal generating set i.e. B spans V and $\forall v \in B$: $B \setminus \{v\}$ does not generate V.

3° B is a maximal linearly indep. set, i.e. B is lin. indep. and $\forall v \in V \setminus B$: $B \cup \{v\}$ is linearly dependent.

<u>Proof</u>: $1° \Rightarrow 2°$ B basis $\Rightarrow$ $\mathrm{Span}(B) = V$

Take $v \in B$. $B_1 := B \setminus \{v\}$.
Then $v \notin \mathrm{Span}(B_1)$ as

B is lin. independent.
$\Rightarrow \mathrm{Span}(B_1) \neq V$.

$2° \Rightarrow 1°$ From 2° we have $\mathrm{Span}(B) = V$
B must be lin indep. Otherwise we can remove an elt. and still span V.

$1^0 \Leftrightarrow 3^0$ Exercise $\Leftarrow$

<u>Notation:</u> We write $W \leq V$ for saying that W is a subspace of V.

<u>Example 181:</u> Is $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

a basis of $\mathbb{R}^4$?

Check linear independence.

$$A := \begin{pmatrix} 1 & 1 & -1 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 3 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

So $A \lambda^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ has only the zero solution.

$\Rightarrow \mathcal{B}$ is linearly independent.

Why is $\mathcal{B}$ a basis?

<u>Corollary 182:</u> Let V be an n-dimensional vector space and $B \subseteq V$ with $|B| = n$.

Then: (a) If B is linearly independent then B is a basis

(b) If B is a generating set for V. Then B is a basis.

Proof: (a) Take $v \in V \setminus B$

$|B \cup \{v\}| > n$

$\Rightarrow$ $B \cup \{v\}$ is not linearly independent by Lemma 171.

$\Rightarrow$ $v \in \text{Span}(B)$. (Why?)

(b) If B is not a basis, then $\exists\, v \in B$ s.t. $B \setminus \{v\}$ still spans V.

$|B \setminus \{v\}| = n-1 < n = \dim V$

Lemma 171 $\Rightarrow$ Contradiction

$\square$

<u>Theorem 183:</u> (base extension theorem)

Let $\dim V < \infty$ and $T, S \subseteq V$ such that T is linearly independent and S spans V.

Then $\exists$ $B \subseteq V$ a basis: $T \subseteq B \subseteq S$

<u>Proof:</u> Exercise $\square$

<u>Example 184:</u>

$W = \operatorname{Span}(S)$,

$$S := \left\{ \begin{array}{cccccc} v_1 & v_2 & v_3 & v_5 & v_6 & v_4 \\ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, & \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, & \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, & \begin{pmatrix} 4 \\ 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, & \begin{pmatrix} -2 \\ 3 \\ 2 \\ -1 \\ 1 \end{pmatrix} \end{array} \right\}$$

Find a maximal linear indep. subset T of S.

(Such a T must be a basis of W, because of theorem 183:

Take a basis B: $T \subseteq B \subseteq S$. B is lin. indep and T is maximal in $S \Rightarrow T = \mathcal{B}$.)

Form a matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & -2 & 1 & 4 \\ 0 & 1 & -1 & 3 & 0 & 1 \\ 1 & 0 & -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow[\text{tions}]{\text{row opera-}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 1 & 0 & 1 \\ & & 1 & -2 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & \end{pmatrix}$$

r.r.e.f.

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 2 & -2 & 1 & 4 \\ 0 & 1 & -1 & 3 & 0 & 1 \\ 0 & -2 & -3 & 4 & -1 & -3 \\ 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & -3 & -3 & 3 & -2 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 2 & -2 & 1 & 4 \\ & 1 & -1 & 3 & 0 & 1 \\ & & -5 & 10 & -1 & -1 \\ & & -2 & 4 & 0 & 0 \\ & & -6 & 12 & -2 & -2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 4 & -8 & 1 & 2 \\ & 1 & -1 & 3 & 0 & 1 \\ & & 1 & -2 & 0 & 0 \\ & & 0 & 0 & 1 & 1 \\ & & & & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 1 & 0 & 1 \\ & & 1 & -2 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & \end{pmatrix}$$

(i) Thus λ_4 and λ_6 are free variables

$\Rightarrow$ We can remove v_4 and v_6.

(ii) $\lambda_1, \lambda_2, \lambda_3, \lambda_5$ are the pivot variables

$\Rightarrow T = \{v_1, v_2, v_3, v_5\}$ is linearly independent.

$\Rightarrow$ T is max lin. indep in S, because of (i).

$\Rightarrow T = \{v_1, v_2, v_3, v_5\}$ is a basis of W.

<u>Def 185:</u> Let $A \in \mathbb{R}^{m \times n}$.

Consider the columns $c_j(A) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} \in \mathbb{R}^m$

and rows $r_i(A) = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^{1 \times n}$

(i) $\mathrm{col}(A) := \mathrm{Span}\{c_j(A) \mid j \in \{1, \ldots, n\}\} \subseteq$

is called the <u>column space</u> of A

$\mathrm{row}(A) := \mathrm{Span}(\{r_i(A) \mid 1 \leq i \leq m\}) \subseteq \mathbb{R}^{1\times n}$

is called the <u>row space</u> of A.

$\mathrm{sol}(A, | \begin{smallmatrix}0\\ \vdots \\ 0\end{smallmatrix})) = \{x \in \mathbb{R}^{n\times 1} \mid Ax = 0\}$

is called the <u>null space</u> of A
Write null (A).

(ii) The dimension of ~~row(A)~~ col(A) is called the <u>rank</u> of A
Write $\mathrm{rk}(A)$.

The dim. of null(A) is called the <u>nullity</u> of A. Write <u>nullity(A)</u>

<u>Example 186:</u> (a) $A = I_n \in \mathbb{R}^{n\times n}$

$$\mathrm{col}(A) = \mathrm{Span}(\{e_1, \ldots, e_n\}) = \mathbb{R}^{n\times 1}$$

$$\Rightarrow \mathrm{rk}(A) = n$$

$$\mathrm{null}(A) = \{\underline{0}\} \Rightarrow \mathrm{nullity}(A) = 0.$$

(b) (see Example 184)

$W = \mathrm{col}(A)$ $\quad \mathrm{rk}(A) = 4$

$$\mathrm{null}(A) = \left\{ \begin{pmatrix} -\lambda_6 \\ -\lambda_4 - \lambda_6 \\ 2\lambda_4 \\ \lambda_4 \\ -\lambda_6 \\ \lambda_6 \end{pmatrix} \;\middle|\; \lambda_4, \lambda_6 \in \mathbb{R} \right\}$$

$$= \mathrm{Span}\Bigg(\underbrace{\left\{ \begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}_{C} \Bigg)$$

C is a basis because every elt of null(A) has unique coordinates w.r.t. C.

$\Rightarrow$ C is a basis of null(A)
$\Rightarrow$ nullity(A) = 2.

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<u>Theorem 187:</u> Let $A \in \mathbb{R}^{m \times n}$

(a) $rk(A) = \dim col(A) = \dim row(A)$

(b) The pivot columns of A form a basis of $col(A)$

(c) $rk(A) = \#$ pivot columns of A
$= \#$ steps in the r.r.e.f. of A
nullity$(A) = \#$ non-pivot columns of A

(d) $rk(A) +$ nullity$(A) = \#$ columns $= n$.

"Dimension theorem for matrices".

<u>Proof:</u> Let R be the reduced row echelon form of A.

Row operations do not change the row space.

$\Rightarrow$ $row(A) = row(R)$, $R = \left(\quad \right)$

Non-zero rows of R are linearly independent.

$\Rightarrow$ $\dim \text{row}(A) = \#$ non-zero rows of R
$= \#$ pivot columns of A

$= \dim \text{col}(A). = \text{rk}(A)$
$\uparrow$
(Exercise! See Example 184)

(b) See Example 184. Fill the proof!

(c) $\text{rk}(A) \overset{(b)}{=} \#$ pivot columns of A.

$\text{rk}(A) \overset{(a)}{=} \dim \text{row}(A)$
$= \#$ non-zero rows in R

$\text{nullity}(A) = \#$ free variables of $Ax=$
$\uparrow$
(See Example 186(b))

$= \#$ non-pivot columns of A.

(d) $n = (\#$ pivot columns$) + (\#$ non-pivot columns$)$

$\underset{(c)}{=}$ $\text{rk}(A) + \text{nullity}(A)$

<u>Example 188</u>: (a) $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 4}$

$rk(A) = 2$, because A is in row echelon form and has 2 non-zero rows.

$\text{nullity}(A) = 4 - rk(A) = 4 - 2 = 2$

(b) $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 4}$

$$A \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- 3-steps $\Rightarrow$ $rk(A) = 3$
  $\text{nullity}(A) = 4 - 3 = 1.$

- pivot columns are 1, 2, 3.

$\Rightarrow \text{col}(A) = \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ \vdots \\ 0 \end{pmatrix} \right\}\right)$

$\text{null}(A) = \left\{ \begin{pmatrix} +\lambda_4 \\ 0 \\ -\lambda_4 \\ \lambda_4 \end{pmatrix} \middle| \lambda_4 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Date

<u>Prop 189:</u> Let V be f.d. and $W \leq V$. Then $\dim W \leq \dim V$.

<u>Proof:</u> Exercise □

<u>Prop 190:</u> Let V be f.d. and $W_1, W_2 \leq V$. Then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

<u>Proof:</u> Exercise on Problem sheet 8. □

IV 3. Transition matrix for base change

Prop. 191: Let V be f.d. and $B \subseteq V$ be a basis of V. ($B = \{v_1, \dots, v_m\}$)
Take $v \in V$.
Then $\exists! \lambda_1, \dots, \lambda_m \in \mathbb{R}$ $v = \lambda_1 v_1 + \dots + \lambda_m v_m$.
(v is a linear combination of $v_1, \dots, v_m$ in a unique way.)

Proof: $\cdot$ $\mathrm{Span}(B) = V \Rightarrow \exists \lambda_1, \dots, \lambda_m \in \mathbb{R}$: $v = \lambda_1 v_1 + \dots + \lambda_m v_m$.

$\cdot$ Let $\mu_1, \dots, \mu_m \in \mathbb{R}$ such that $v = \mu_1 v_1 + \dots + \mu_m v_m$

Then $0_V = (\lambda_1 - \mu_1) v_1 + \dots + (\lambda_m - \mu_m) v_m$

B is linearly independent
$\Rightarrow \lambda_1 = \mu_1$ and ... and $\lambda_m = \mu_m$. □

We call $\lambda_1, \dots, \lambda_m$ the coordinates of v w.r.t. $B = \{v_1, \dots, v_m\}$ and write

$$[v]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$$

"the coordinate vector of v with respect to B".

<u>Caution:</u> For $[v]_B$ we have fixed an order for the elements of B!

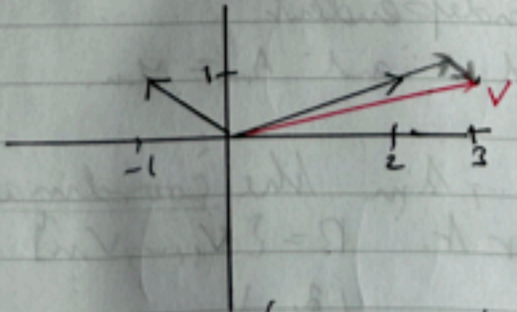
So here B is an ordered set.

<u>Example 192:</u> $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \mathbb{R}^2$

(a) $B_1 = \{e_1, e_2\}$ $\qquad [v]_{B_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

because $v = 3e_1 + 1 \cdot e_2$

(b) $B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$



$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \lambda_1 = \frac{4}{3}, \quad \lambda_2 = -\frac{1}{3}, \quad [v]_{B_2} = \begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix}$$

(c) What is $[e_1]_{B_2}$ and $[e_2]_{B_2}$?

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mu_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = I_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}.$

Thus: $\begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

$$\Rightarrow \quad [e_1]_{B_2} = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}, \; [e_2]_{B_2} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

(d) We have

$$3e_1 + 1 \cdot e_2 = v$$

In B_1 coo: $3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

In B_2 coo: $\underbrace{3 \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix} + 1 \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}}_{\left([e_1]_{B_2}, \, [e_1]_{B_2}\right) \cdot [v]_{B_1}} = \begin{pmatrix} 4/3 \\ -2/3 \end{pmatrix}$

$$\left([e_1]_{B_2}, [e_1]_{B_2}\right) \cdot [v]_{B_1} = [v]_{B_2}$$

We call $P_{(B_1 \to B_2)} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

the transition matrix (for the base change) from B_1 to B_2

Idea: You plug in B_1 - coor. and get B_2 - coor.

plug in $\underset{\begin{pmatrix} 3 \\ 1 \end{pmatrix}}{\overset{\shortparallel}{[v]_{B_1}}}$ and get $\underset{\begin{pmatrix} 4/3 \\ -\frac{1}{3} \end{pmatrix}}{\overset{\shortparallel}{[v]_{B_2}}}$

Example: Find $\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{B_2}$.

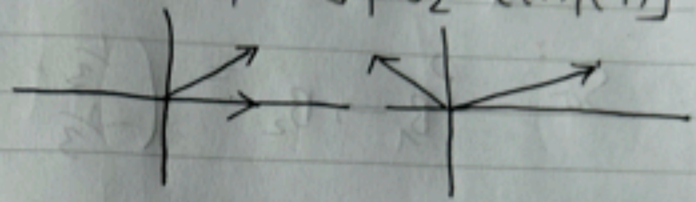
$$\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{B_2} = P_{(B_1 \to B_2)} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{B_1}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Let us play a bit!

<u>Example 193:</u> What about other base changes? $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, $B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$



We want $P_{(B_1 \to B_2)}$.

We just have $P_{(B_1 \to B_{st})} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and

$$P_{(B_2 \to B_{st})} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then $P_{(B_1 \to B_2)} = P_{(B_{st} \to B_2)} \circ P_{(B_1 \to B_{st})}$

(1st change from B_1 to B_{st}, then from B_{st} to B_2)

$$= P^{-1}_{(B_2 \to B_{st})} \circ P_{(B_1 \to B_{st})}.$$

Idea $\left(P_{B_2 \to B_{st}} \middle| P_{B_1 \to B_{st}} \right) \xrightarrow[\text{row}]{} \left(I_n \middle| P_{B_1 \to B_2} \right)$

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$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 1/3 & 2/3 \\ 0 & 1 & -1/3 & 1/3 \end{array}\right)$$

So $P_{B_1 \to B_2} = \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$

<u>Notation 194:</u> $B = \{v_1, \dots, v_m\}$, $C = \{w_1, \dots, w_m\}$ bases for V.

$$P_{B \to C} := \left([v_1]_C, \dots, [v_m]_C \right)$$

is called the transition matrix from B to C.

We have $P_{B \to C} \circ [v]_B = [v]_C$ for v e

<u>Example 195:</u> $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

$C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Bases? Yes! (Their determinants are non-zero.)

$P_{B \to C}$?

$$\left(P_{C \to B_{st}} \;\middle|\; P_{B \to B_{st}} \right) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 & 1 & 2 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -5 & 1 & -3 & -2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & \frac{3}{5} & \frac{3}{5} \\ & 1 & & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ & & 1 & & -1 & \end{array}\right)$$

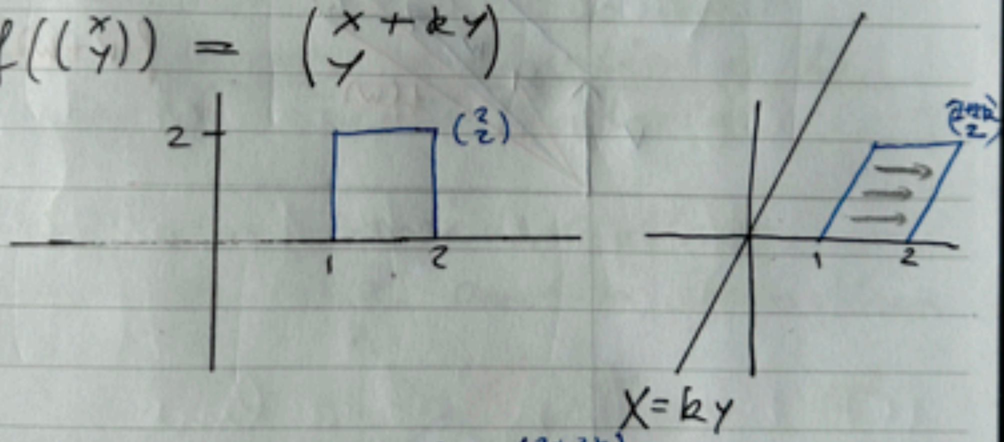
$$\underbrace{\qquad\qquad}_{P_{B \to C}}$$

IV.4 Geometry of matrix transformations.

<u>Motivation 196:</u> (shearing)

Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map which translates a point $\binom{x}{y}$ parallel to the x-axis by amount ky. ($k \in \mathbb{R}$ fixed.)

$$f\left(\binom{x}{y}\right) = \binom{x+ky}{y}$$



$$f\left(\binom{2}{2}\right) = \binom{2+2k}{2}$$

<u>Observe:</u> (a) f is determined by
- fixing $\binom{1}{0}$ and
- moving $\binom{0}{1}$ to $\binom{k}{1}$

$$f\left(\binom{x}{y}\right) = x\binom{1}{0} + y\binom{k}{1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\binom{x}{y}$$

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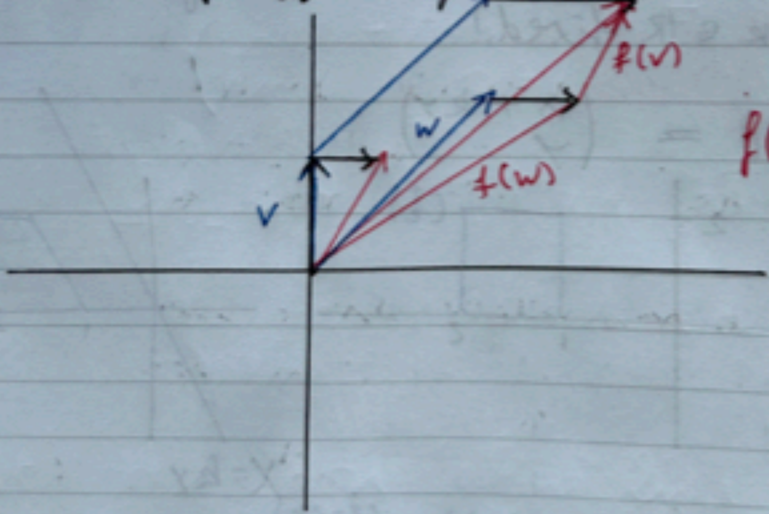
(e) f respects linear combinations.

$$f\left(\lambda\begin{pmatrix}v_1\\v_2\end{pmatrix}+\mu\begin{pmatrix}w_1\\w_2\end{pmatrix}\right)=\begin{pmatrix}\lambda v_1+\mu w_2+k(\lambda v_2+\mu w_2)\\\lambda v_2+\mu w_2\end{pmatrix}$$

$$=\lambda\begin{pmatrix}v_1+kv_2\\v_2\end{pmatrix}+\mu\begin{pmatrix}w_1+kw_2\\w_2\end{pmatrix}$$

$$=\lambda f\left(\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)+\mu\left(f\left(\begin{pmatrix}w_1\\w_2\end{pmatrix}\right)\right).$$



$f(v+w)=f(v)+f(w)$

<u>Def 197:</u> Let V, W be vector spaces

A map $f: V\longrightarrow W$ is called linear if
- $\forall_{\lambda\in\mathbb{R}}\ \forall_{v\in V}: f(\lambda v)=\lambda f(v)$
- $\forall_{v_1,v_2\in V}: f(v_1+v_2)=f(v_1)+f(v_2)$

f is called a matrix transformation if $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ (some n,m) and $\exists_A \in \mathbb{R}^{m\times n}$:

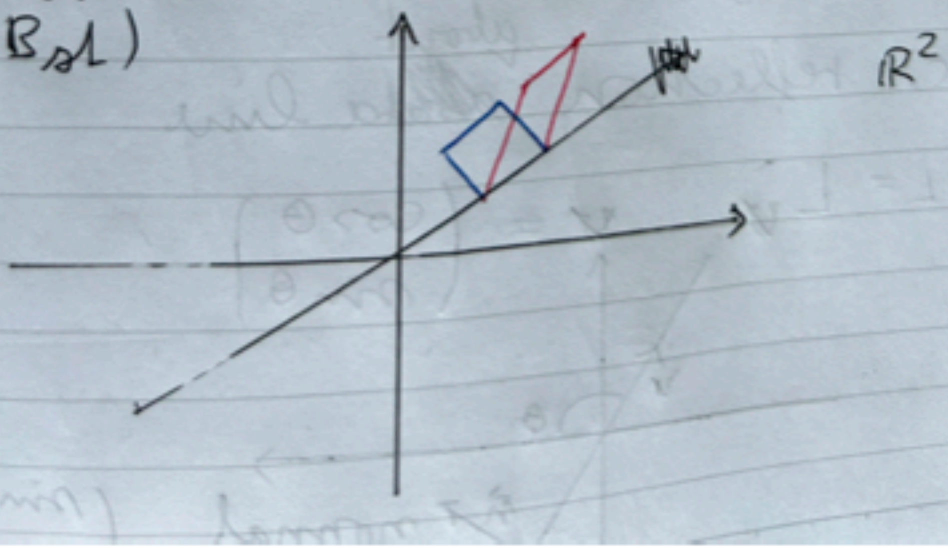
$$f(v) = Av \quad \forall v \in \mathbb{R}^n \triangleq \mathbb{R}^{n\times 1}.$$

In this case we denote f by T_A.

<u>Examples 198:</u> (for $\mathbb{R}^2$)

(a) f in Motivation 196 is called a <u>shearing in direction of the x-axis</u> with factor k.

You can consider other coordinates (i.e. another basis than B_{st})



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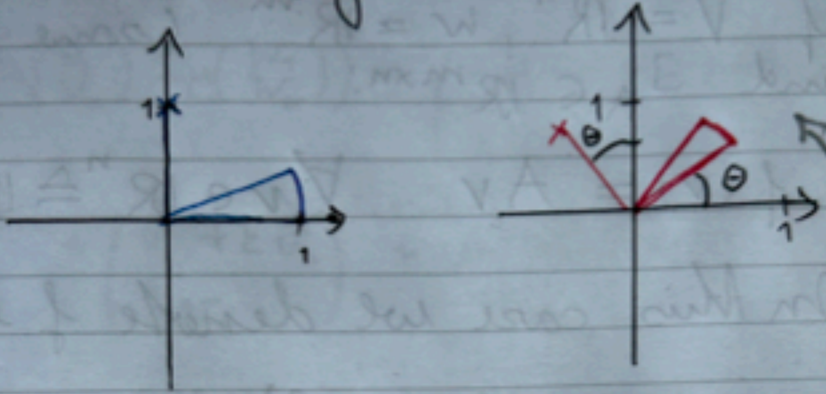
(b) rotation by $\theta \in [0, 2\pi[$.



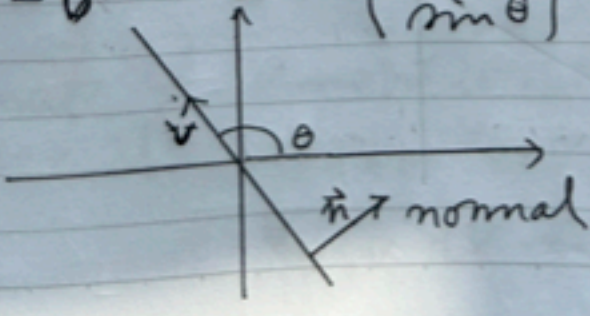
$$T_{\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}} =: R(\theta)$$

$\underline{\text{E.g.}}$ for $\theta = \pi$ we get $T_{\begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}}$

the point reflection at the origin

(c) reflection ~~at~~ about a line

$$L = L_{\mathbf{v}} \qquad \mathbf{v} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$



$$\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} \cos(\theta - 90^\circ) \\ \sin(\theta - 90^\circ) \end{pmatrix} = \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix}$$

$$T\underbrace{\begin{pmatrix} \end{pmatrix}}_{?} =: S_L$$

In $B := \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \right\}$ - coordinates

we have

$$\left[S_L(w)\right]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [w]_B.$$

$$S_L(v) = v, \quad S_L(\vec{n}) = -\vec{n}.$$

(Notation: $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = [S_L]_B$

The <u>matrix for</u> S_L <u>relative to B.</u>)

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We want $[S_L]_{B_{\mathcal{N}}}$.

$\ulcorner$ We have $P_{B \to B_{\mathcal{N}}} [S_L]_B [w]_B = \cancel{[\ ]}_{B_{\mathcal{N}}}$

$= [S_L(w)]_{B_{\mathcal{N}}}$

i.e. $P_{B \to B_{\mathcal{N}}} [S_L]_B P_{B_{\mathcal{N}} \to B} \underbrace{P_{B \to B_{\mathcal{N}}} [w]_B}_{[w]_{\mathcal{N}}}$

$= [S_L(w)]_{\mathcal{N}}$ $\lrcorner$

So $[S_L]_{\mathcal{N}} = P_{B \to B_{\mathcal{N}}} [S_L]_B P_{B_{\mathcal{N}} \to B}$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

So $S_L = T_{\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}}$

For $\theta = 0$ we get $S_L = T_{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}$

is the reflection about the x-axis.

(d) What is $T_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$?

("orthogonal projection onto the x-axis.")

Homework: Learn about
- contraction / dilation
- extension / compression.

Exercise 199: Let $v \in \mathbb{R}^{2 \times 1}$ of norm 1.

Let P_v be the orthogonal projection onto L_v. Find

A such that $P_v = T_A$.

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What happens in $\mathbb{R}^3$?

Example 200:

(a) shearing:

$$\begin{pmatrix} 1 & k & \\ & 1 & \\ & & 1 \end{pmatrix}$$ and similar

matrices (similar means conjugate)

Above matrix: translation of a point in $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ direction proportional to its $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ distance to the x-z plane.

(b) rotation around a line (called "rotation axis")

Ex: around x-axis:

$$T_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}}$$

In general $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^{3\times 1}$ $\|v\|_2 = 1$

W.l.o.g. $v_1 \neq 0$.

$R_v(\theta)$ = rotation around L_v by angle θ.

Find A ?

Step 1: find $\overbrace{\{v, v', v''\}}^{B}$ an orthonormal basis, i.e.

- $\|v\|_2 = \|v'\|_2 = \|v''\|_2 = 1$

and

- $v \cdot v' = v \cdot v'' = v' \cdot v'' = 0$

We also want a right handed basis, i.e.

$$\det \left([v]_{B_{st}} \; [v']_{B_{st}} \; [v'']_{B_{st}} \right) > 0.$$

Take $v' = \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{v_1^2 + v_2^2}}$

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$$V'' = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ -v_2 & v_1 & 0 \end{vmatrix} \frac{1}{\sqrt{v_1^2+v_2^2}}$$

$$= \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_1^2 + v_2^2 \end{pmatrix} \frac{1}{\sqrt{v_1^2+v_2^2}}$$

$$P_{B \to B_{st}} = \begin{pmatrix} v_1 & -v_2/\sqrt{\ } & -v_1 v_3/\sqrt{\ } \\ v_2 & v_1/\sqrt{\ } & -v_2 v_3/\sqrt{\ } \\ v_3 & 0 & \sqrt{\ } \end{pmatrix}$$

$$\left[R_v(\theta)\right]_B = \begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}$$

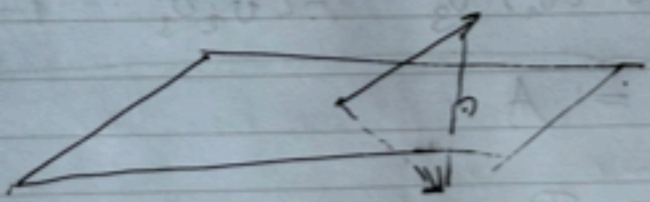
$$\left[R_v(\theta)\right]_{B_{st}} = P_{B \to B_{st}} \begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix} P_{B_{st} \to B}$$

(c) reflection about a (plane / line / pt)

We only consider plane here.

Ex: E = x-y plane

$$\text{Proj}_E = T_{\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}}$$



In general: $E = H_v$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ non-zero.

W.l.o.g. $\|v\|_2 = 1$.

Formula: $\text{Proj}_{H_v}(w) = w - 2(w \bullet v)v$

(Think about how to describe the reflection about a line.)

$$[\mathrm{Proj}_{H_v}]_{B_{st}} = \Big([\mathrm{Proj}_{H_v}(e_1)]_{st}, [\mathrm{Proj}_{H_v}(e_2)]_{st}, [\mathrm{Proj}_{H_v}(e_3)]_{st} \Big)$$

$$= \underbrace{\begin{pmatrix} (1-2v_1)v_1 & -2v_2v_1 & -2v_3v_1 \\ (0-2v_1)v_2 & 1-2v_2v_2 & -2v_3v_2 \\ (0-2v_1)v_3 & -2v_2v_3 & 1-2v_3v_3 \end{pmatrix}}_{=:\,A}$$

$$\mathrm{Proj}_{H_v} = T_A.$$

<u>Terminology 201</u>: Let $f : \mathbb{R}^n \to \mathbb{R}^u$ be a linear map.

Then $[f]_{st} = \big([f(e_1)]_{st}, \dots, [f(e_n)]_{st} \big)$ is called the standard matrix of f.

Remark 202: If A is the standard matrix of f then $f = J_A$.

Proof: $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

$$f(v) = \sum v_i f(e_i) = \sum v_i [f(e_i)]_{\mathcal{M}}$$

$$= Av \qquad \square$$

Def 203: Let V be a v.s.

A subset of the form $v + W$ with $v \in V$ and $W \leq V$ is called an affine subspace.

If $\dim V < \infty$, then we call $\dim W$ the dimension of $v+W$. Put $\dim(v+W) := \dim W$.

Prop 204: Let $f: V \longrightarrow W$ be linear.

Then 1) f maps an affine subspace onto an affine subspace, e.g. an affine line onto a point or an affine line.

2) The <u>kernel</u> of f

$\ker(f) := \{ v \in V \mid f(v) = 0_W \}$ is a subspace of V.

and if $\dim V < \infty$ we have $\dim V = \dim \ker(f) + \dim \operatorname{im}(f)$

"Dimension theorem for linear maps"

<u>Proof</u>: 1) $v + U$ an affine subspace of V.

$f(v+U) = f(v) + f(U)$

$f(U) \leq W$ (subspace criterium)

So $f(v+U)$ is a subspace.

Case $L_{v'} = U$: $f(v + L_{v'}) = f(v) + \mathbb{R} f(v')$
a pt. or an affine line.

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2) Take $B = \{v_1, \dots, v_n\}$ basis of V such that $C = \{f(v_1), \dots, f(v_r)\}$ is a basis of $\mathrm{im}(f) =: U$.

Represent $f: V \longrightarrow U$ by a matrix

$$A = [f]_{C,B}$$
$$= \left([f(v_1)]_C, \dots, [f(v_n)]_C\right)$$
$$= \begin{pmatrix} 1 & & 0 & \\ & \ddots & & * \\ 0 & & 1 & \end{pmatrix}$$

$[f]_{C,B}$ is called the matrix for f relative to the bases B and C

Then $\mathrm{null}(A) = \{[v]_B \mid v \in \ker(f)\}$

Thus $\dim \mathrm{null}(A) = \dim \ker(f)$

$$\overset{\shortparallel}{n-r} = \dim V - \dim(\mathrm{im}(f))$$

$\square$

Later we will prove:

Theorem 205:

Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map s.t. $\|f(v)\| = \|v\|$ for all $v \in \mathbb{R}^3$.

Suppose $\det [f]_{st} = 1$.

Then f is a rotation.

Example 206:

$$f = T_A \quad \text{for} \quad A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$f(v) = \begin{pmatrix} \frac{v_1}{\sqrt{2}} + \frac{v_2}{\sqrt{2}} \\ -v_3 \\ -\frac{v_1}{\sqrt{2}} + \frac{v_2}{\sqrt{2}} \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- $\|f(v)\|^2 = \frac{1}{2}(v_1 + v_2)^2 + v_3^2 + \frac{1}{2}(v_2 - v_1)^2$
$= v_1^2 + v_2^2 + v_3^2 = \|v\|^2$

$\cdot f$ is linear

$\Rightarrow$ f is a rotation by Theorem 205.

Find: 1) line of rotation
 2) rotation angle.

1) Solve $\quad Av = v$

$\Leftrightarrow \quad (A - I_3)v = 0$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} - 1 & \frac{1}{\sqrt{2}} & \\ & -1 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\text{sol}^n = \{(\lambda, \lambda(\sqrt{2} - 1), \lambda(1 - \sqrt{2}))^T \mid \lambda \in \mathbb{R}\}$

is the line of rotation.

$$v^{(1)} := \begin{pmatrix} 1 \\ \sqrt{2} - 1 \\ 1 - \sqrt{2} \end{pmatrix}$$

$$v^{(2)} := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \perp v^{(1)}.$$

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$$f(v^{(2)}) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\Rightarrow \cos\theta = \left(v^{(2)} \cdot f(v^{(2)})\right) \frac{1}{\|v^{(2)}\| \, \|f(v^{(2)})\|}$$

$$= \left(\frac{1}{\sqrt{2}} - 1\right) \frac{1}{2}$$

$$= \frac{1-\sqrt{2}}{2\sqrt{2}}$$

θ is $98.42°$ or $-98.42°$

Which one?

Check right hand ~~rule~~ rule

$$\begin{vmatrix} 1 & \sqrt{2}-1 & 1-\sqrt{2} \\ & 1 & 1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$\underset{\substack{\uparrow \\ 2^{nd}\text{ row}}}{=} \begin{vmatrix} 1 & 1-\sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} - \begin{vmatrix} 1 & \sqrt{2}-1 \\ \frac{1}{\sqrt{2}} & -1 \end{vmatrix}$$

$$= 1 - \left(-1 - \left(1 - \frac{1}{\sqrt{2}}\right)\right) = \frac{-1}{\sqrt{2}} + 3 > 0$$

$\Rightarrow$ θ is $98,42°$