

Chapter IV

General vector spaces.

Def 151: A triple $(V, +, \cdot)$ consisting of a set V and maps

$$+ : V \times V \rightarrow V \quad (\text{addition})$$

$$\cdot : \mathbb{R} \times V \rightarrow V \quad (\text{scalar multiplication})$$

is called real vector space if it satisfies the axioms of Prop 106, i.e.

$$(1a) \text{ (ass, +), (0, +), (inv, +)}$$

$$(1b) \text{ (com, +)}$$

$$(2) \text{ (ldist, \cdot, +), (rdist, +_{\mathbb{R}}, \cdot)}$$

$$\text{(as, } |\mathbb{R}| \text{)}, \text{ (unit, \cdot)}$$

Remark 152: (1a) means that $(V, +)$ is a "group".

(1a) and (1b) means that $(V, +)$ is an "abelian group".

So a vector space is a triple $(V, +, \cdot)$ s.t.
 $(V, +)$ is an abelian group which satisfies (2).

Example 153:

(a) $(\mathbb{R}, +, \cdot)$ is a real vector space because $(\mathbb{R}, +)$ is an abelian group and

(2) • (ldist) and (rdist) are satisfied, because $(+, \cdot)$ satisfies distributivity

- $(a \cdot b, \cdot, \cdot)$ is satisfied as $\cdot$ is associative
- $1_{\mathbb{R}} \cdot a = a \quad \forall a \in \mathbb{R}$.

(b) Consider $\{0\} \subseteq \mathbb{R}$.

$(\{0\}, +, \cdot)$ is a vector space

as $0+0=0$ and $1 \cdot 0 = 0 \quad \forall \lambda \in \mathbb{R}$

(c) $\mathbb{R}^{2 \times 2}$. The space of 2 times 2 matrices is a vector space w.r.t.

$$+ : \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$$

$$\cdot : \mathbb{R} \times \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$$

$$A + B := \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$\lambda A := \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}$$

$(\mathbb{R}^{2 \times 2}, +)$ is an abelian group

by

$$(a_{ij}, +) : (A + B) + C = ((a_{ij} + b_{ij}) + c_{ij})_{ij}$$

$$\stackrel{(a_{ij}, +_{\mathbb{R}})}{=} \underset{\uparrow}{(a_{ij} + (b_{ij} + c_{ij}))}_{ij} = A + (B + C)$$

$$(0, +) : 0 + A = (0 + a_{ij})_{ij} \stackrel{(0, +_{\mathbb{R}})}{=} (a_{ij})_{ij}$$

$$= (a_{ij} + 0)_{ij} = A + 0$$

$$(0 = (\underset{\mathbb{R}}{0})_{ij})$$

(inv, +)

For A take $B := (-a_{ij})_{i,j}$ (add inv to R)

$$\text{Then } A + B = (a_{ij} + (-a_{ij}))_{i,j}$$

$$= 0 = ((-a_{ij}) + a_{ij})_{i,j}$$

$$(\text{com}, +) \quad A + B = (a_{i,j} + b_{i,j})$$

$$\stackrel{\uparrow}{=} (b_{i,j} + a_{i,j}) = B + A$$

(com, +)

(2) is satisfied too:

$$\lambda(A + B) \stackrel{\text{Def.}}{=} (\lambda(a_{i,j} + b_{i,j}))$$

$$= (\lambda a_{i,j} + \lambda b_{i,j}) \stackrel{\text{Def.}}{=} (\lambda a_{i,j}) + (\lambda b_{i,j})$$

$$\stackrel{\uparrow}{=} \lambda A + \lambda B$$

Def.

$$(\lambda_1 + \lambda_2)A \stackrel{\text{Def.}}{=} ((\lambda_1 + \lambda_2)a_{i,j})$$

$$\stackrel{\uparrow}{=} (\lambda_1 a_{i,j} + \lambda_2 a_{i,j}) \stackrel{\text{Def.}}{=} (\lambda_1 a_{i,j}) + (\lambda_2 a_{i,j})$$

(dist, R)

Def.

$$\stackrel{\uparrow}{=} \lambda_1 A + \lambda_2 A$$

Def.

$$(\lambda_1 \lambda_2)A \stackrel{\text{Def.}}{=} ((\lambda_1 \lambda_2)a_{i,j}) \stackrel{\text{Def.}}{=} (\lambda_1(\lambda_2 a_{i,j}))$$

(ass, R)

$$\begin{array}{c} \overline{\lambda_1} \\ \uparrow \\ \text{Def.} \end{array} \quad \lambda_1 (\lambda_2 a_{ij})_{ij} = \lambda_1 (\lambda_2 A) \quad \begin{array}{c} \uparrow \\ \text{Def.} \end{array}$$

$$\cdot \quad 1_R A = \begin{array}{c} \uparrow \\ \text{Def.} \end{array} (1_R a_{ij})_{ij} = \begin{array}{c} \uparrow \\ (\text{unit}, \cdot_R) \end{array} (a_{ij})_{ij} = A.$$

(d) $\text{Map}(\mathbb{N}, \mathbb{R})$ = set of sequences with real terms

$$(f+g)(n) := f(n) + g(n)$$

$$(\lambda f)(n) := \lambda f(n)$$

$(\text{Map}(\mathbb{N}, \mathbb{R}), +)$ is an abelian group, because

$(\text{comm}, +)$ and $(\text{ass}, +)$ follow from $(\text{comm}, +_{\mathbb{R}})$ and $(\text{ass}, +_{\mathbb{R}})$.

For the neutral element take $0: \mathbb{N} \rightarrow \mathbb{R}$, defined as $0(n) := 0_{\mathbb{R}}$

The additive inverse of f is

$$g: \mathbb{N} \rightarrow \mathbb{R}, \quad g(n) := -f(n).$$

(2) The axioms (ldist), (rdist),
(ass, i_1) and (Unit, $\cdot$) are
shown as for $\mathbb{R}^{2 \times 2}$.

(e) A strange example.

$$V := \mathbb{R}^{>0}$$

$$+_V : V \times V \rightarrow V, a +_V b := a \cdot_{\mathbb{R}} b$$

$$\cdot_V : \mathbb{R} \times V \rightarrow V, \lambda \cdot_V a := a^\lambda$$

$$(a, b \in V = \mathbb{R}^{>0}, \lambda \in \mathbb{R})$$

(1) $(V, +_V)$ is an abelian
group (Exercise)

What is the neutral element?

"

add. inverse of $v \in V$?

(2) (E dist)

$$\lambda \cdot_V (a +_V b) = \lambda \cdot_V (a \cdot_{\mathbb{R}} b) = (a \cdot b)^\lambda$$

$$\stackrel{\uparrow}{=} a^\lambda b^\lambda = (\lambda \cdot_V a) \cdot_{\mathbb{R}} (\lambda \cdot_V b)$$

(additivity
of taking a power)

$$= (\lambda \cdot v)a + v(\lambda \cdot v)b$$

(rdist), (asn, $\mathbb{R}$, v), (unit, $\cdot$)
are exercises.

Def 154: (subspace)

Let $(V, +, \cdot)$ be an $\mathbb{R}$ -vector space

We call a subset $W \subseteq V$ a subspace of V if

$(W, +|_{W \times W}, \cdot|_{\mathbb{R} \times W})$ is a
vector space.

(in particular $+(W \times W) \subseteq V$
and $\cdot(\mathbb{R} \times W) \subseteq W$)

Example 155 If $(V, +, \cdot)$ is a real
vector space then $\{0_V\}$ is
a subspace of V .

Pf: We only have to show

$$(i) + (\{0_V\} \times \{0_V\}) \subseteq \{0_V\}$$

$$(ii) \cdot (\mathbb{R} \times \{0_V\}) \subseteq \{0_V\}.$$

(iii) and 0_V is the add inverse of 0_V .

$$(i) \quad 0_V + 0_V \underset{(0, +)}{=} 0_V$$

$$(ii) \quad \lambda 0_V \underset{(0, +)}{=} \lambda (0_V + 0_V) \underset{(\text{ldist})}{=} \lambda 0_V + \lambda 0_V$$

$$\Rightarrow \quad 0_V = \lambda 0_V$$

$+ (- (\lambda 0_V))$

(iii) By (i) we have $0_V + 0_V = 0_V$.

$$\text{so } -0_V = 0_V. \quad \square$$

$$(b) \quad \mathbb{R} \times \{0\} = \{(r, 0) \mid r \in \mathbb{R}\}$$

is a subspace of $(\mathbb{R}^2, +, \cdot)$

$$(+ : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, \cdot : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2)$$

because $\text{im} \left(+ \Big|_{(\mathbb{R} \times \{0\}) \times (\mathbb{R} \times \{0\})} \right)$

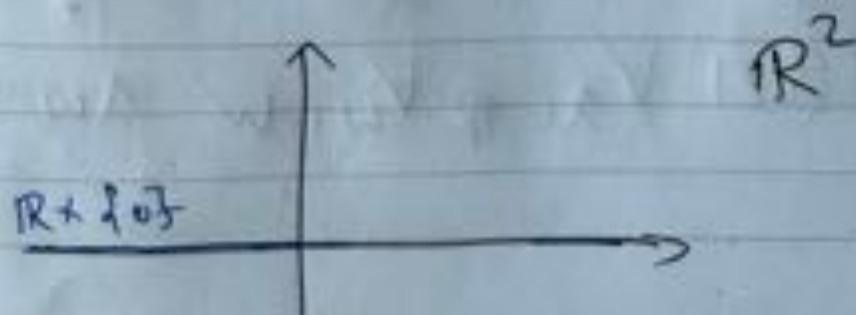
$$\subseteq \mathbb{R} \times \{0\}$$

$$\left((r_1, 0) +_{\mathbb{R}^2} (r_2, 0) = (r_1 + r_2, 0 + 0) = (r_1 + r_2, 0) \right. \\ \left. \in \mathbb{R} \times \{0\} \right)$$

$$\text{and } \text{im} \left(\cdot \Big|_{\mathbb{R} \times (\mathbb{R} \times \{0\})} \right) \subseteq \mathbb{R} \times \{0\}$$

$$\left(\lambda_{\mathbb{R}^2}(r, 0) = (\lambda r, 0) \in \mathbb{R} \times \{0\} \right)$$

and the next Lemma.



(c) $\mathbb{R} \times \{1\} = \{(r, 1) \mid r \in \mathbb{R}\}$
 is not a subspace of $(\mathbb{R}^2, +, \cdot)$,
 because

$$(1, 1) + (0, 1) = (1, 2) \notin \mathbb{R} \times \{1\}.$$

(d) $\mathbb{Q} \subseteq (\mathbb{R}, +, \cdot)$ is not a
 (real) subspace ~~of~~, because

$$\sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Q}, \text{ i.e.}$$

$$\text{im}(\cdot /_{\mathbb{R} \times \mathbb{Q}}) \not\subseteq \mathbb{Q}.$$

Lemma 156: Let $(V, +_V, \cdot_V)$ be
 a vector space. $\emptyset \neq W \subseteq V$ is a
 subspace of V iff

$$(S1) \quad \forall w_1, w_2 \in W: w_1 +_V w_2 \in W$$

$$(S2) \quad \forall \lambda \in \mathbb{R} \forall w \in W: \lambda w \in W$$

In that case we have

$$0_W = 0_V \text{ and}$$

The additive inverse of w in $(W, +_W)$ is the " " " " w in $(W, +_W)$

Proof: Part 1: The equivalence.

" $\Rightarrow$ " $\checkmark$ by definition of a subspace

" $\Leftarrow$ " To show $(W, +_W, \cdot_W)$ is a vector space.

$(W, +_W)$ is an abelian group:

$(\text{com}, +_W)$ and $(\text{ass}, +_W)$ are inherited from $+_V$.

$(0, +_W)$: Take $w_0 := 0_V$. Then for $w \in W$

$$w_0 + w = w + w_0 = w.$$

We have to show $w_0 \in W$.

$$W \neq \emptyset \Rightarrow \exists w \in W$$

$$(S2) \Rightarrow w_0 = 0_V = \underset{\uparrow}{0_R} \cdot_V w \in W$$

Exercise

$(inv, +_W)$ Take $w \in W$.

$$\text{We have } \underbrace{(-1) \cdot_V w}_{=: \tilde{w}} \in W$$

$$\text{and } \tilde{w} +_V w = \underset{\uparrow}{\tilde{w} +_V 1 \cdot_V w} \\ \text{(unit, } \cdot \text{)}$$

$$= (-1) \cdot_V w + 1 \cdot_V w$$

$$\underset{\uparrow}{=} \underset{\text{(\&dist)}}{((-1) +_R 1) \cdot_V w} = \underset{\uparrow}{0_R \cdot_V w}$$

(-1) add inverse
of 1 in $(R, +)$

$$\begin{matrix} \leftarrow \\ \uparrow \end{matrix} \quad 0_V = w_0$$

Exercise

(2) $(r\text{-dist}), (l\text{-dist}), (a\text{-dist}), (u\text{-dist}), (v\text{-dist})$ are inherited from $(V, +, \cdot, \cdot)$.

This also shows the second part of the Lemma.

(Note: neutral elt and add. inverse are unique.)

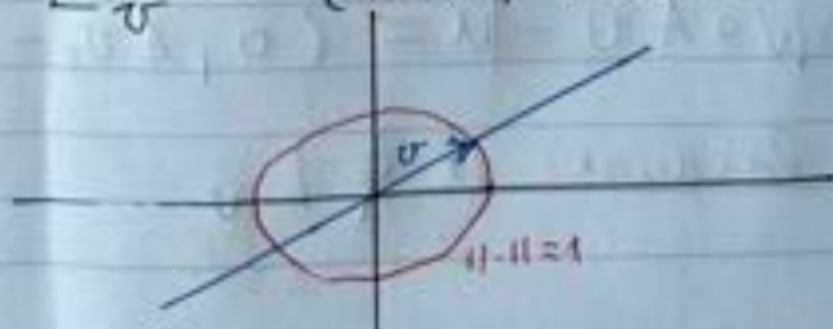
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Example 157: All subspaces of $(\mathbb{R}^2, +, \cdot)$:

• $\{0\}, \mathbb{R}^2$, Else?

• Take $v \in \mathbb{R}^2$ with $\|v\|=1$

$$L_v = \{tv \mid t \in \mathbb{R}\}$$



Any more? No!

Proof: Let $W \subseteq \mathbb{R}^2$ be a non-zero subspace.

Take $w \in W$ non-zero.

Then $v = \frac{1}{\|w\|} \cdot w \in W$ by (S2).

$\Rightarrow L_v \subseteq W$ by (S2)

To show $W = L_v$.

Assume $W \neq L_v$.

Take $u \in W \setminus L_v$.

W.l.o.g. $v_1 \neq 0$ ($v = (v_1, v_2)$)

$\Rightarrow \exists \lambda \in \mathbb{R}: \lambda v_1 = u_1$ ($u = (u_1, u_2)$)

$\Rightarrow \exists \lambda v - u = (0, \lambda v_2 - u_2) \neq 0$

because $u \notin L_v$

Thus $e_2 = (0, 1) \in W$, by (S2)

and $e_1 = (1, 0) = \frac{1}{v_1} (v - v_2 e_2)$

$\in W$

Thus $\forall x = (x_1, x_2) \in \mathbb{R}^2$ we have

$$x = x_1 e_1 + x_2 e_2 \in W, \text{ i.e. } W = \mathbb{R}^2$$

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□

Exercise 158: What about all subspaces in $\mathbb{R}^3$, or better all subspaces of $\mathbb{R}^3$ containing

the line $L_{(1,1,0)}$?

Example 159: $W := (1, 1) + L_{(1,1,0)} \subseteq \mathbb{R}^2$

is not a subspace. (Exercise.)

General constructions of subspaces

Prop 160: Let V be a vector space

and $W_1, \dots, W_k$ be subspaces. Then

(a) The intersection $W_1 \cap \dots \cap W_k$ is a subspace of V

(b) The sum

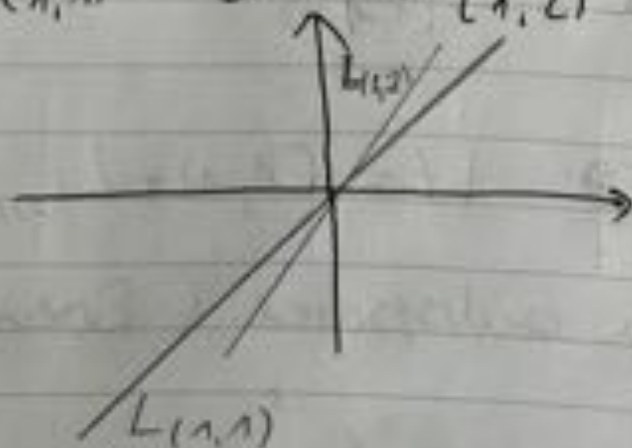
$$W_1 + \dots + W_k := \{w_1 + \dots + w_k \mid w_i \in W_i, i=1, \dots, k\}$$

is a subspace of V .

Pt: Problem sheet 7. $\square$

Example 161:

(i) $L(1,1)$ and $L(1,2)$ in $\mathbb{R}^2$



$$L(1,1) \cap L(1,2) = \{(0,0)\} \text{ zero space}$$

$$L_{(1,1)} + L_{(1,2)} = \{ \lambda(1,1) + \mu(1,2) \mid \lambda, \mu \in \mathbb{R} \} \\ = \mathbb{R}^2$$

(for $(x_1, x_2) \in \mathbb{R}^2$, take $\mu = x_2 - x_1$
and $\lambda = 2x_1 - x_2$, i.e.

$$(x_1, x_2) = (2x_1 - x_2) \cdot (1, 1) + (x_2 - x_1)(1, 2).$$

(ii) In $\mathbb{R}^4$. Consider the planes

$$E_1 = \{ \lambda(1, 0, 0, 0) + \mu(1, 1, 1, 1) \mid \lambda, \mu \in \mathbb{R} \}$$

$$E_2 = \{ \lambda(1, 0, 0, 0) + \mu(1, 1, 0, 0) \mid \lambda, \mu \in \mathbb{R} \}$$

$$E_1 \cap E_2 = \mathbb{R}(1, 0, 0, 0)$$

Pf: " $\supseteq$ " ✓ " $\subseteq$ " Take $v \in E_1 \cap E_2$

$$\text{Then } v = \lambda(1, 0, 0, 0) + \mu(1, 1, 1, 1) \\ = \lambda(1, 0, 0, 0) + \mu(1, 1, 0, 0)$$

for some $\lambda, \mu, \alpha, \beta \in \mathbb{R}$.

$$\Rightarrow (\lambda - \alpha, 0, 0, 0) = (\mu - \alpha, \mu - \beta, -\beta, -\beta)$$

$$\Rightarrow \lambda = \alpha \text{ and } \mu = \alpha = 0.$$

$$\Rightarrow v \in \mathbb{R}(1, 0, 0, 0)$$

□

$$H = E_1 + E_2 = \left\{ \lambda(1, 0, 0, 0) + \mu(1, 1, 1, 1) + u(1, 1, 0, 0) \mid \lambda, \mu, u \in \mathbb{R} \right\}$$

We will see that this is a linear hyperplane (later).

Let's compute a normal

$$\vec{e}_1 = (1, 0, 0, 0), \vec{e}_2 = (0, 1, 0, 0), \vec{e}_3, \vec{e}_4$$

$$\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 0 \cdot \vec{e}_1 - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \vec{e}_2$$

$$+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \vec{e}_3 - \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} \vec{e}_4$$

$$= 0 \vec{e}_1 - 0 \vec{e}_2 + (-1) \vec{e}_3 - (-1) \vec{e}_4$$

$$= (0, 0, -1, 1)$$

So we get $H = H_{(0,0,-1,1)}$,

i.e. the linear hyperplane perpendicular to $(0,0,-1,1)$.

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IV 2 Basis of a vector space

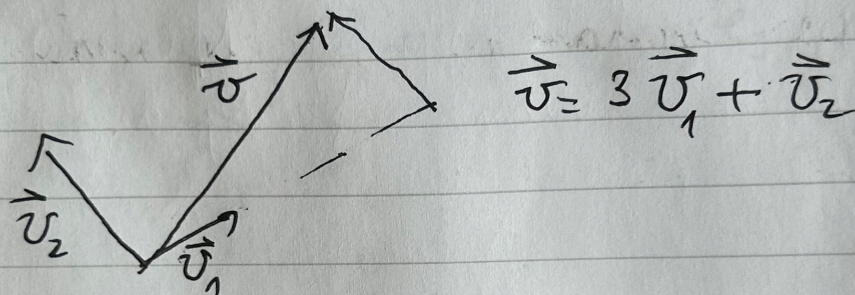
Let V be a vector space.

Def 162: Take $\vec{v}, \vec{v}_1, \dots, \vec{v}_m \in V$.

We say that $\vec{v}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_m$ if $\exists \lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$\vec{v} = \lambda_1 \vec{v}_1 + \dots + \lambda_m \vec{v}_m$$

Picture:



Motivation for basis 163:

$$\begin{aligned} W &= \mathbb{R} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mathbb{R} \begin{pmatrix} -1 \\ 3 \\ -3.5 \end{pmatrix} + \mathbb{R} \begin{pmatrix} 5 \\ -9 \\ 14.5 \end{pmatrix} \\ &= \left\{ \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 3 \\ -3.5 \end{pmatrix} + \lambda_4 \begin{pmatrix} 5 \\ -9 \\ 14.5 \end{pmatrix} \right\} \\ &\quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \end{aligned}$$

Part 1: Describe W with a small set of vectors.

Easy: Take $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -3.5 \end{pmatrix}, \begin{pmatrix} 5 \\ -9 \\ 14.5 \end{pmatrix} \right\}$

Part 2: We want to have the set as small as possible, i.e. no vector in the set is a linear combination of the other vectors.

Step 1: We see $\begin{pmatrix} 5 \\ -9 \\ 14.5 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 3 \\ -3.5 \end{pmatrix}$

$\Rightarrow$ We can remove $\begin{pmatrix} 5 \\ -9 \\ 14.5 \end{pmatrix}$.

$$S_1 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -3.5 \end{pmatrix} \right\}$$

Every vector $\in W$ is a lin. combination of vectors in S_1 .

Step 2: Solve $\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 3 \\ -3.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Take $\lambda_1 = 1.5, \lambda_2 = -2.5, \lambda_3 = -1$

We can ~~then~~ remove any of them.

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$$S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 3.5 \end{pmatrix} \right\}$$

Step 3: Solve

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 3 \\ 3.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \lambda_2 = 0 = \lambda_1$$

So we cannot make S_2 smaller.

Part 1 $\Rightarrow$ Concept of generating set

Part 2 $\Rightarrow$ Concept of linear independence

IV 2.1. Generating set

Prop 163: Let $S \subseteq V$. We put

$$U_1 := \bigcap_{\substack{W \supseteq S \\ W \text{ subspace of } V}} W \quad \text{and}$$

$$U_2 := \left\{ \lambda_1 v_1 + \dots + \lambda_m v_m \mid m \in \mathbb{N}_0, \right. \\ \left. \lambda_1, \dots, \lambda_m \in \mathbb{R} \text{ and } v_1, \dots, v_m \in S \right\}$$

Then U_1 and U_2 are subspaces of V
and $U_1 = U_2$.

Proof: Subspaces: Use subspace criterion
(Lemma 156)

For U_1 exercise

For U_2 : $U_2 \neq \emptyset$, because $0 \in U_2$
(0 is the empty sum)

$$\lambda \in \mathbb{R}, \lambda_1 v_1 + \dots + \lambda_m v_m \in U_2$$

$$\Rightarrow \lambda (\lambda_1 v_1 + \dots + \lambda_m v_m) =$$

$$(\lambda \lambda_1) v_1 + \dots + (\lambda \lambda_m) v_m \in U_2$$

$$\lambda_1 v_1 + \dots + \lambda_m v_m, \mu_1 w_1 + \dots + \mu_n w_n \in U_2$$

$$\Rightarrow (\lambda_1 v_1 + \dots + \lambda_m v_m) + (\mu_1 w_1 + \dots + \mu_n w_n)$$

$\in U_2$, because all $v_i, w_j \in S$
and $\lambda_i, \mu_j \in \mathbb{R}$.

Equality: " $\subseteq$ " U_2 subspace of V , $U_2 \supseteq S$

$$\Rightarrow U_1 \subseteq U_2$$

" $\supseteq$ " Take $\lambda_1 v_1 + \dots + \lambda_m v_m \in U_2$

and $W \supseteq S$ a subspace. Then
as $v_1, \dots, v_m \in W$ we get

$$\lambda_1 v_1 + \dots + \lambda_m v_m \in W$$

So

$$- " - \in \bigcap_{\substack{W \text{ subspace} \\ W \supseteq S}} W = U_1$$

Terminology 164:

$\text{Span}(S) := \bigcap_{\substack{W \supseteq S \\ \text{subspace of } V}} W$ is called

the span of S

If $U = \text{Span}(S)$ then we say
that S spans U .

Example 165

$$(a) \quad H_{(0,0,1)} = \{(x_1, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}$$

$$= \mathbb{R} \underset{\substack{\text{"} \vec{v}_1 \text{"} \\ (1,0,0)}}{\vec{e}_1} + \mathbb{R} \underset{\substack{\text{"} \vec{v}_2 \text{"} \\ (0,1,0)}}{\vec{e}_2} = \text{Span}(\{\vec{e}_1, \vec{e}_2\})$$

$$(e) \quad H_{(1,1,1)} = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$$

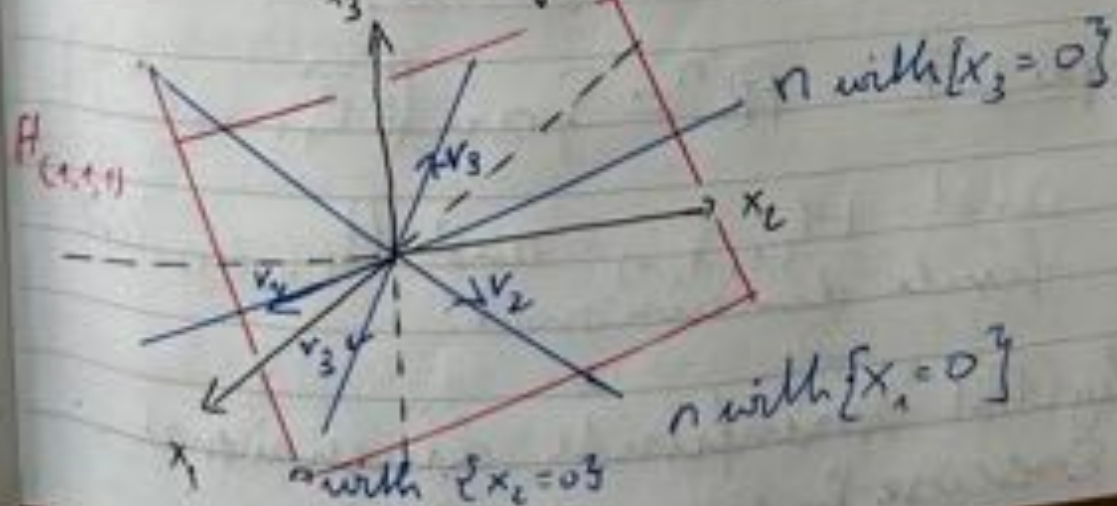
$$\supseteq S := \left\{ \underset{\substack{\text{"} \vec{v}_1 \text{"} \\ (1,-1,0)}}{(1,-1,0)}, \underset{\substack{\text{"} \vec{v}_2 \text{"} \\ (0,1,-1)}}{(0,1,-1)}, \underset{\substack{\text{"} \vec{v}_3 \text{"} \\ (1,0,-1)}}{(1,0,-1)} \right\}$$

$$\text{Span}(S) = H_{(1,1,1)}$$

Proof: " $\subseteq$ " ✓
 " $\supseteq$ " Take $v = (x_1, x_2, x_3) \in H_{(1,1,1)}$

$$\Rightarrow v = x_1(1,-1,0) + (x_1 + x_2)(0,1,-1)$$

$$\in \text{Span}(S) \quad \square$$



(c) $\text{Poly}(\mathbb{R}) := \{f: \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ polynomial function}\}$

{ a function of the form
 $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$
 with $a_0, \dots, a_n \in \mathbb{R}$ }

Put $P_i(x) := x^i$

$\Rightarrow \text{Poly}(\mathbb{R}) = \text{Span}(\{P_0, P_1, P_2, \dots\})$

Def 166: V is called finite-dimensional (or finitely generated)

if $\exists S \subseteq V$ finite: $\text{Span}(S) = V$

otherwise V is called
infinite dimensional

Example 167: $\mathbb{R}^n = \text{Span}(\{\vec{e}_1, \dots, \vec{e}_n\})$

and $H_{(1,1,1,1)}$ and $H_{(0,0,1)}$ are
 finite dimensional.

$\text{Poly}(\mathbb{R})$ is infinite dimensional
 (Exercise in example class)

IV 2.2. Linear independence

Def 168: A family of vectors $v_1, \dots, v_m \in V$ is called linearly independent if the equation

$$\lambda_1 v_1 + \dots + \lambda_m v_m = 0_V$$

has only the trivial solution, i.e.

$$\lambda_1 = \dots = \lambda_m = 0_{\mathbb{R}}$$

A ^{finite} set $S \subseteq V$, say $S = \{v_1, \dots, v_m\}$ with $v_i \neq v_j$ for $i \neq j$, is called linearly independent if the family $v_1, \dots, v_m$ is linearly independent.

Otherwise they are called linearly dependent.

Rk 169: Linear indep for ∞ sets, see problem sheet.

Example 170: (a) in Motivation 163.
 S is not linearly independent.
 S_1 — " —

S_2 is linearly independent.

(b) $\{0_V\}$ is linearly dependent,
as $1_{\mathbb{R}} \cdot 0_V = 0_V$.

(c) $v \in V \setminus \{0_V\}$.

$S = \{v\}$ is linearly independent.

Pr:

$\lambda v = 0_V$. Assume $\lambda \neq 0$

$$\Rightarrow 0_V = \lambda^{-1}(\lambda v) = (\lambda^{-1}\lambda) v = 1_{\mathbb{R}} v = v$$

(d) $\vec{e}_i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^n$

$S = \{\vec{e}_1, \dots, \vec{e}_n\}$ is linearly indep.

Pr: $\sum_{i=1}^n \lambda_i \vec{e}_i = \underline{0}$

$$\Rightarrow (\lambda_1, \lambda_2, \dots, \lambda_n) = \underline{0}$$

$$\Rightarrow \lambda_1 = \dots = \lambda_n = 0 \quad \square$$

(e) See 165(b)

$$S = \left\{ \underset{v_1}{(1, -1, 0)}, \underset{v_2}{(0, 1, -1)}, \underset{v_3}{(1, 0, -1)} \right\}$$

is linearly dep. as

$$v_1 + v_2 - v_3 = 0 \in \mathbb{R}^3$$

$$(f) \quad S = \{f_0, f_1, f_2\} \subseteq \text{Map}(\mathbb{R}, \mathbb{R})$$

$$f_0(x) = 1$$

$$f_1(x) = \cos x$$

$$f_2(x) = \cos(2x) \quad , x \in \mathbb{R}$$

Then S is lin. indep.

$$\text{Pf: } \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 = 0 \text{ - zero function.}$$

$$\text{Plug in: } x=0: \lambda_0 + \lambda_1 + \lambda_2 = 0$$

$$x = \frac{\pi}{2}: \lambda_0 - \lambda_2 = 0$$

$$x = \frac{\pi}{4}: \lambda_0 + \frac{1}{\sqrt{2}} \lambda_1 = 0$$

$$x = \pi: \lambda_0 - \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \lambda_0 = \lambda_1 = \lambda_2 = 0$$

□

$$(g) \quad f_3: \mathbb{R} \rightarrow \mathbb{R}, \quad f_4(x) = \sin^2 x$$

$S = \{f_0, f_1, f_2, f_3\}$ is lin. dependent

$$\text{Pf: } \cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

$$\Rightarrow 0 = f_0 - 2f_3 - f_2$$

□

IV 2.3. BasisLemma 171: Suppose $V = \text{Span}(S)$ with $|S| < \infty$ and let $T \subseteq V$ be finite and linearly independent.

Then

$$|T| \leq |S|.$$

Proof: Assume $|T| > |S|$

$$T = \{v_1, \dots, v_n\}, S = \{w_1, \dots, w_m\}, m < n$$

$$\text{Write } v_j = \sum_{i=1}^m a_{ij} w_i, \quad a_{ij} \in \mathbb{R}.$$

$$\text{Solve } \lambda_1 v_1 + \dots + \lambda_n v_n = 0_V, \text{ e.g.}$$

$$\text{by } A \lambda^T = 0, \text{ for } A = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n.$$

Then

$$A \xrightarrow{\text{row operations}}$$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

r.r.e.f.

$m < n \Rightarrow \exists$ non-pivot column
 $\Rightarrow \exists$ free variable λ_{i_0}

$\Rightarrow \exists$ solution $\lambda \in \text{sol}(A, 0)$
 with $\lambda_{i_0} = 1$

$\Rightarrow v_1, \dots, v_n$ are not lin. indep. $\nsubseteq \square$
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Ex Def 172: $\nexists$ a lin. indep. set $T \subseteq \mathbb{R}^3$

with more than 3 elements, because
 $\text{Span}(\{e_1, e_2, e_3\}) = \mathbb{R}^3$

Def 173: (basis) Let V be finite dim.
 and $B \subseteq V$ finite.
 We call B a basis of V if
 B is linearly independent and
 $\text{Span}(B) = V$.

Plk 174: If V is infinite dimensional
 $B \subseteq V$ is called a basis if
 B is lin. indep. and spans V .

Example 175: (a) See 163: $S_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -4,5 \end{pmatrix} \right\}$

is a basis of W .

(b) $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$
 "standard basis of $\mathbb{R}^n$ "

(c) $S = \{(1, -1, 0), (0, 1, -1)\}$
 is a basis of $H_{(1,1,1)}$ (See 165 (b))
 because S is generating $H_{(1,1,1)}$
 and linearly indep.

$$\lambda_1(1, -1, 0) + \lambda_2(0, 1, -1) = 0$$

$$\Rightarrow \lambda_1 = 0 \text{ and } \lambda_2 = 0.$$

$\{(1, -1, 0), (1, 0, -1)\}$ is also a basis.

(d) $\{p_0, p_1, p_2, \dots\}$ is a basis
 for $\text{Poly}(\mathbb{R})$. (an example
 for an infinite dimensional
 vector space.)

Theorem 176: Let V be finite dimen-
 sional and B_1, B_2 be bases for V .
 Then $|B_1| = |B_2|$.

Proof: B_1 generates V and B_2 is linearly independent $\Rightarrow |B_2| \leq |B_1|$ by Lemma 171.

$|B_2| \leq |B_1|$ similarly. $\square$

Def 177: Let V be finite dim. and $B \subseteq V$ a basis.

$\dim V := |B|$ is called the dimension of V .

Ex 178: $\dim \mathbb{R}^3 = 3$

$$\dim H_{(1,1,1)} = 2$$

$$\text{In 163: } \dim W = 2$$

$$\dim \{0_V\} = 0$$

$$\dim L_v = 1, v \in V \setminus \{0\}.$$

$$(L_v = \mathbb{R}v)$$

Example 179: The set

$T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right\}$ is linearly independent.

because

$$\text{sol}\left(\left(\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)\right) = \{0\}$$

$$\text{and } |T| = 3 = \dim \mathbb{R}^3.$$

Claim: T spans $\mathbb{R}^3$

Pf: If not, take $v \in \mathbb{R}^3 - \text{Span}(T)$

Then $T \cup \{v\}$ is still linearly independent, because
a non-zero solⁿ of

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v = 0$$

must have $\lambda_4 \neq 0$ (why?)
Thus $v \in \text{Span}(T)$ ∇ .

$$\text{Lemma 171} \Rightarrow |T \cup \{v\}| \leq 3 \nabla$$

□

More general:

Theorem 180: Let V be f.d. and

$B \subseteq V$ be a finite subset. T.q.e.:

1° B is a basis of V

2° B is a minimal generating set
i.e. B spans V and

$\forall v \in B : B - \{v\}$ does not
generate V .

3° B is a maximal linearly indep.
set, i.e. B is lin. indep. and

$\forall v \in V \setminus B : B \cup \{v\}$ is linearly
dependent.

Proof: $1^\circ \Rightarrow 2^\circ$ B basis $\Rightarrow \text{Span}(B) = V$

Take $v \in B$. $B_1 := B - \{v\}$.

Then $v \notin \text{Span}(B_1)$ as

B is lin. independent.

$\Rightarrow \text{Span}(B_1) \neq V$.

$2^\circ \Rightarrow 1^\circ$ From 2° we have $\text{Span}(B) = V$

B must be lin indep. Otherwise
we can remove an elt. and still span V .

$1^{\circ} \Leftrightarrow 2^{\circ}$ Exercise $\square$

Notation: We write $W \leq V$ for saying that W is a subspace of V .

Example 181: Is $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

a basis of $\mathbb{R}^4$?

Check linear independence.

$$A := \begin{pmatrix} 1 & 1 & -1 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 3 & -5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

So $A A^T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has only the zero solution.

$\Rightarrow B$ is linearly independent.

Why is B a basis?

Corollary 182: Let V be an n -dimensional vector space and $B \subseteq V$ with $|B| = n$.

Then: (a) If B is linearly independent then B is a basis.

(b) If B is a generating set for V , then B is a basis.

Proof: (a) Take $v \in V \setminus B$.
 $|B \cup \{v\}| > n$

$\Rightarrow B \cup \{v\}$ is not linearly independent by Lemma 171.

$\Rightarrow v \in \text{Span}(B)$. (Why?)

(b) If B is not a basis, then $\exists v \in B$ s.t. $B \setminus \{v\}$ still spans V .

$|B \setminus \{v\}| = n-1 < n = \dim V$
 Lemma 171 $\Rightarrow$ Contradiction $\square$

Theorem 183: (base extension theorem)

Let $\dim V < \infty$ and $T \cup S \subseteq V$ such that T is linearly independent and S spans V .

Then $\exists B \subseteq V$ a basis: $T \subseteq B \subseteq S$

Proof: Exercise $\square$

Example 184:

$$W = \text{Span}(S),$$

$$S = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 2 \\ -1 \\ -1 \\ -1 \end{pmatrix}}, \overset{v_5}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}}, \overset{v_6}{\begin{pmatrix} 4 \\ 1 \\ 3 \\ -1 \end{pmatrix}}, \overset{v_4}{\begin{pmatrix} -2 \\ 3 \\ 2 \\ -1 \end{pmatrix}} \right\}$$

Find a maximal linear indep. subset of S .

(Such a T must be a basis of W , because of Theorem 183. ~~Trigger~~)

Take a basis B : $T \cup B \subseteq S$. B is lin. indep.

and T is maximal in $S \Rightarrow T=B$)

Form a matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & -2 & 1 & 4 \\ 0 & 1 & -1 & 3 & 0 & 1 \\ 1 & 0 & -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ & & 1 & -2 & 0 & 0 \\ & & & & 1 & 1 \end{pmatrix}$$

r.r.e.f.

$$A \rightarrow \begin{pmatrix} 1 & 2 & 2 & -2 & 14 \\ 0 & 1 & -1 & 3 & 0 & 1 \\ 0 & -2 & -3 & 4 & -1 & -3 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & -3 & -3 & 3 & -2 & -5 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 2 & -2 & 14 \\ & 1 & -1 & 3 & 0 & 1 \\ & & -5 & 10 & -1 & -1 \\ & & -2 & 4 & 0 & 0 \\ & & -6 & 12 & -2 & -2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 4 & -8 & 12 \\ & 1 & -1 & 3 & 0 & 1 \\ & & 1 & -2 & 0 & 0 \\ & & 0 & 0 & 1 & 1 \\ & & & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 0 & 1 & 0 & 1 \\ & & 1 & -2 & 0 & 0 \\ & & & 1 & 1 & \end{pmatrix}$$

(i) Thus λ_4 and λ_6 are free variables.

$\Rightarrow$ We can remove v_4 and v_6 .

(ii) $\lambda_1, \lambda_2, \lambda_3, \lambda_5$ are the pivot variables.

$\Rightarrow T = \{v_1, v_2, v_3, v_5\}$ is linearly independent.

$\Rightarrow T$ is max lin. indep in S , because of (i).

$\Rightarrow T = \{v_1, v_2, v_3, v_5\}$ is a basis of W .

Def 185: Let $A \in \mathbb{R}^{m \times n}$.

Consider the columns $c_j(A) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix} \in \mathbb{R}^m$

and rows $r_i(A) = (a_{i1}, \dots, a_{in}) \in \mathbb{R}^{1 \times n}$

(i) $\text{col}(A) := \text{Span} \{c_j(A) \mid j \in \{1, \dots, n\}\}$

is called the column space of A

$$\text{row}(A) := \text{Span}(\{r_i(A) \mid 1 \leq i \leq m\}) \subseteq \mathbb{R}^{1 \times n}$$

is called the row space of A .

$$\text{col}(A, \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) = \{x \in \mathbb{R}^{n \times 1} \mid Ax = 0\}$$

is called the null space of A .
Write $\text{null}(A)$.

- (ii) The dimension of ~~row(A)~~ $\text{col}(A)$ is called the rank of A .
Write $\text{rk}(A)$.

The dim. of $\text{null}(A)$ is called the nullity of A . Write nullity(A).

Example 186: (a) $A = I_n \in \mathbb{R}^{n \times n}$

$$\text{col}(A) = \text{Span}(\{e_1, \dots, e_n\}) = \mathbb{R}^{n \times 1}$$

$$\Rightarrow \text{rk}(A) = n$$

$$\text{null}(A) = \{0\} \Rightarrow \text{nullity}(A) = 0.$$

(b) (see Example 184)

$$w = \text{col}(A) \quad \text{rk}(A) = 4$$

$$\text{null}(A) = \left\{ \begin{pmatrix} -\lambda_6 \\ -\lambda_4 - \lambda_6 \\ 2\lambda_4 \\ \lambda_4 \\ -\lambda_6 \\ \lambda_6 \end{pmatrix} \mid \lambda_4, \lambda_6 \in \mathbb{R} \right\}$$

$$= \text{Span} \left(\underbrace{\left[\begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right]}_C \right)$$

C is a basis because every elt of $\text{null}(A)$ has unique coordinates w.r.t. C .

$\Rightarrow C$ is a basis of $\text{null}(A)$

$\Rightarrow \text{nullity}(A) = 2$

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Theorem 187: Let $A \in \mathbb{R}^{m \times n}$

- (a) $\text{rk}(A) = \dim \text{col}(A) = \dim \text{row}(A)$
- (b) The pivot columns of A form a basis of $\text{col}(A)$
- (c) $\text{rk}(A) = \#$ pivot columns of A
 $= \#$ steps in the r.r.e.f. of A
 $\text{nullity}(A) = \#$ non-pivot columns of A
- (d) $\text{rk}(A) + \text{nullity}(A) = \# \text{ columns} = n$

"Dimension Theorem for matrices".

Proof: Let R be the reduced row echelon form of A .

Row operations do not change the row space.

$$\Rightarrow \text{row}(A) = \text{row}(R), \quad R = \begin{pmatrix} \text{---} & 1 & \text{---} & \text{---} \\ & & \ddots & \\ & & & 1 & \text{---} \\ & & & & \text{---} \end{pmatrix}$$

Non-zero rows of R are linearly independent.

$$\Rightarrow \dim \text{row}(A) = \# \text{ non-zero rows of } R \\ = \# \text{ pivot columns of } A$$

$$= \dim \text{col}(A) = \text{rk}(A)$$

↑

(Exercise! See Example 184)

(e) See Example 184. Fill the proof!

$$(c) \text{rk}(A) \stackrel{(b)}{=} \# \text{ pivot columns of } A.$$

$$\text{rk}(A) \stackrel{(a)}{=} \dim \text{row}(A)$$

$$= \# \text{ non-zero rows in } R$$

$$\text{nullity}(A) = \# \text{ free variables of } A\vec{x} =$$

↑

(See Example 186(b))

$$= \# \text{ non-pivot columns of } A.$$

$$(d) \quad n = (\# \text{ pivot columns}) + (\# \text{ non-pivot columns})$$

$$= \text{rk}(A) + \text{nullity}(A)$$

↑
(c)

Example 188: (a) $A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{3 \times 4}$

$\text{rk}(A) = 2$, because A is in row echelon form and has 2 non-zero rows.

$\text{nullity}(A) = 4 - \text{rk}(A) = 4 - 2 = 2$

(b) $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 4}$

$$A \rightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3-steps $\Rightarrow \text{rk}(A) = 3$
 $\text{nullity}(A) = 4 - 3 = 1.$

pivot columns are 1, 2, 3.

$$\Rightarrow \text{col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\text{null}(A) = \left\{ \begin{pmatrix} \lambda_4 \\ 0 \\ -\lambda_4 \\ \lambda_4 \end{pmatrix} \mid \lambda_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Prop 189: Let V be f.d. and $W \leq V$.
Then $\dim W \leq \dim V$.

Proof: Exercise $\square$

Prop 190: Let V be f.d. and $W_1, W_2 \leq V$.
Then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Proof: Exercise on Problem sheet 8. $\square$

IV 3. Transition matrix for base change

NO. 216

Prop. 191: Let V be f.d. and $B \subseteq V$
be a basis of V . ($B = \{v_1, \dots, v_m\}$)
Take $v \in V$.

Then $\exists! \lambda_1, \dots, \lambda_m \in \mathbb{R}$ $v = \lambda_1 v_1 + \dots + \lambda_m v_m$.

(v is a linear combination of $v_1, \dots, v_m$
in a unique way.)

Proof: $\text{Span}(B) = V \Rightarrow \exists \lambda_1, \dots, \lambda_m \in \mathbb{R} : v = \lambda_1 v_1 + \dots + \lambda_m v_m$.

Let $\mu_1, \dots, \mu_m \in \mathbb{R}$ such that $v = \mu_1 v_1 + \dots + \mu_m v_m$.

Then $0_v = (\lambda_1 - \mu_1)v_1 + \dots + (\lambda_m - \mu_m)v_m$

B is linearly independent

$\Rightarrow \lambda_1 = \mu_1$ and $\dots$ and $\lambda_m = \mu_m$. $\square$

We call $\lambda_1, \dots, \lambda_m$ the coordinates
of v w.r.t. $B = \{v_1, \dots, v_m\}$ and
write

$$[v]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$$

"the coordinate vector of v with respect to B ".

Caution: For $[v]_B$ we have fixed an order for the elements of B !

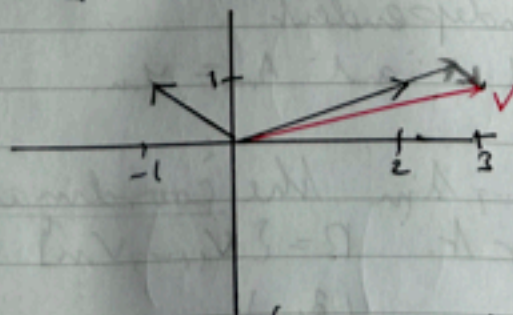
So here B is an ordered set.

Example 192: $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \in \mathbb{R}^2$

$$(a) B_1 = \{e_1, e_2\} \quad [v]_{B_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\text{because } v = 3e_1 + 1 \cdot e_2$$

$$(c) B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$



$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \lambda_1 = \frac{4}{3}, \quad \lambda_2 = -\frac{1}{3}, \quad [v]_{B_2} = \begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix}$$

(c) What is $[e_1]_{B_2}$ and $[e_2]_{B_2}$?

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mu_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{Thus } \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Thus: } \begin{pmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow [e_1]_{B_2} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}, [e_2]_{B_2} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

(d) We have

$$3e_1 + 1 \cdot e_2 = v$$

$$\text{In } B_1 \text{ coord: } 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\text{In } B_2 \text{ coord: } 3 \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} + 1 \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} [e_1]_{B_2} & [e_2]_{B_2} \end{pmatrix} [v]_{B_1} = [v]_{B_2}$$

We call $P_{(B_1 \rightarrow B_2)} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

the transition matrix (for the base change) from B_1 to B_2 .

Idea: You plug in B_1 -coord. and get B_2 -coord.

plug in $\begin{matrix} [v]_{B_1} \\ \text{"} \\ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \end{matrix}$ and get $\begin{matrix} [v]_{B_2} \\ \text{"} \\ \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix} \end{matrix}$

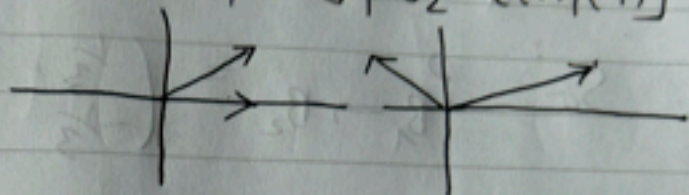
Example: Find $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B_2}$.

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B_2} &= P_{(B_1 \rightarrow B_2)} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{B_1} \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}. \end{aligned}$$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Let us play a bit!

Example 193: What about other base changes? $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, $B_2 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$



We want $P_{(B_1 \rightarrow B_2)}$.

We just have $P_{(B_1 \rightarrow B_{int})} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and

$$P_{(B_2 \rightarrow B_{int})} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$\text{Then } P_{(B_1 \rightarrow B_2)} = P_{(B_{int} \rightarrow B_2)} \circ P_{(B_1 \rightarrow B_{int})}$$

(1st change from B_1 to B_{int} , then from B_{int} to B_2)

$$= P_{(B_2 \rightarrow B_{int})}^{-1} \circ P_{(B_1 \rightarrow B_{int})}.$$

Idea $\left(P_{B_2 \rightarrow B_{int}} \mid P_{B_1 \rightarrow B_{int}} \right) \xrightarrow{\text{row}} \left(I_n \mid P_{B_1 \rightarrow B_2} \right)$

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array} \right)$$

$$\text{So } P_{B_1 \rightarrow B_2} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Notation 194:

$$B = \{v_1, \dots, v_m\},$$

$$C = \{w_1, \dots, w_m\}$$

basis for V .

$$P_{B \rightarrow C} := ([v_1]_C, \dots, [v_m]_C)$$

is called the transition matrix from B to C .

$$\text{We have } P_{B \rightarrow C} \circ [v]_B = [v]_C \text{ for } v \in V$$

Example 195:

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Bases? Yes! (Their determinants are non-zero.)

$P_{B \rightarrow C}$?

$$\left(P_{C \rightarrow B_{\text{std}}} \mid P_{B \rightarrow B_{\text{std}}} \right) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 0 & 1 \\ 3 & 1 & 1 & 0 & 1 & 2 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -5 & 1 & -3 & -2 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & \frac{3}{5} & \frac{3}{5} \\ & 1 & & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ & & 1 & & -1 & 0 \end{array} \right)$$

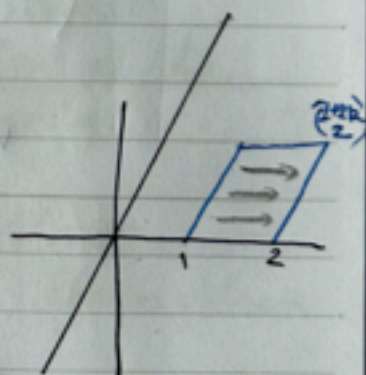
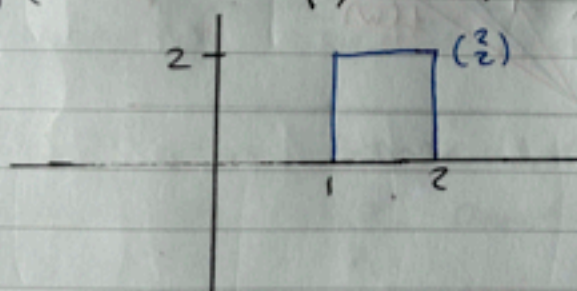
$P_{B \rightarrow C}$

IV 4 Geometry of matrix trans- formations.

Motivation 196: (shearing)

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the map which translates a point $\begin{pmatrix} x \\ y \end{pmatrix}$ parallel to the x -axis by amount ky . ($k \in \mathbb{R}$ fixed.)

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + ky \\ y \end{pmatrix}$$



$$X = ky$$

$$f\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 2+2k \\ 2 \end{pmatrix}$$

Observe: (a) f is determined by

- fixing $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and
- moving $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} k \\ 1 \end{pmatrix}$

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} k \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

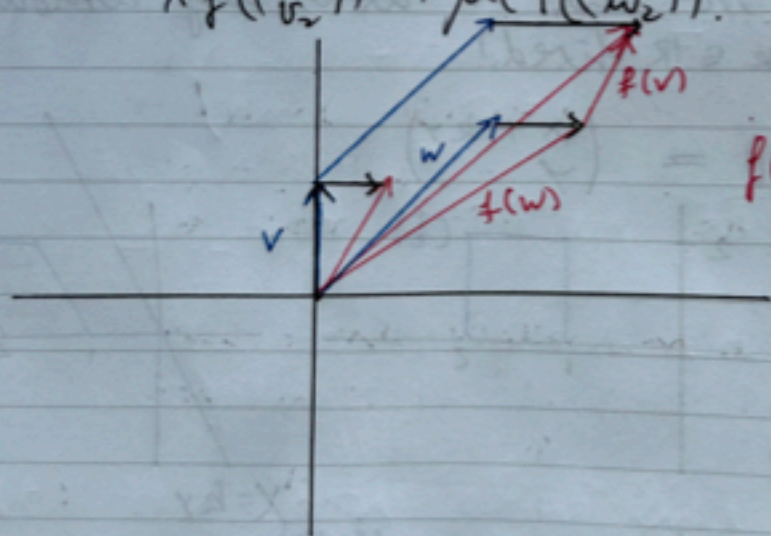
End of Lecture 24th Nov. 23

(c) f respects linear combinations.

$$f\left(\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \mu \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \begin{pmatrix} \lambda v_1 + \mu w_1 + k(\lambda v_2 + \mu w_2) \\ \lambda v_2 + \mu w_2 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} v_1 + k v_2 \\ v_2 \end{pmatrix} + \mu \begin{pmatrix} w_1 + k w_2 \\ w_2 \end{pmatrix}$$

$$= \lambda f\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + \mu f\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right).$$



$$f(v+w) = f(v) + f(w)$$

Def 197: Let V, W be vector spaces

A map $f: V \rightarrow W$ is called

linear if $\forall \lambda \in \mathbb{R} \forall v \in V: f(\lambda v) = \lambda f(v)$

$\forall v_1, v_2 \in V: f(v_1 + v_2) = f(v_1) + f(v_2)$

f is called a matrix transformation
if $V = \mathbb{R}^n$, $W = \mathbb{R}^m$
and $\exists A \in \mathbb{R}^{m \times n}$: (some n, m)

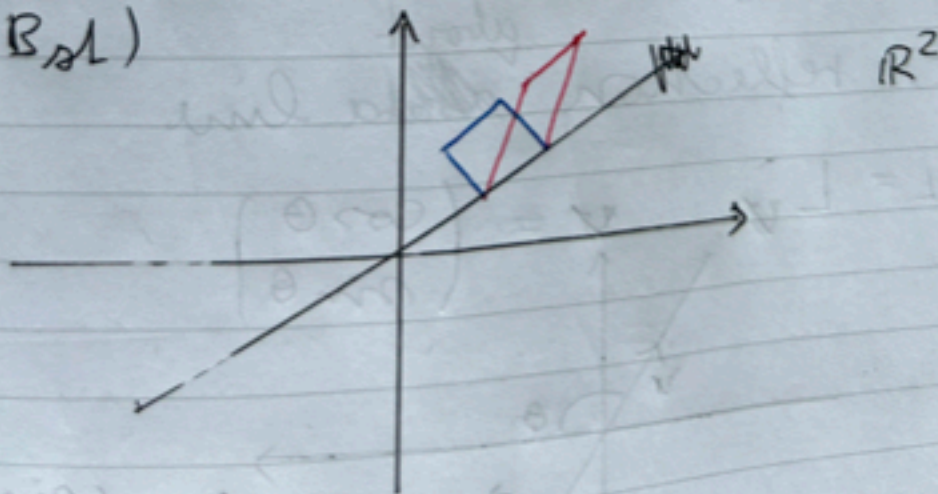
$$f(v) = Av \quad \forall v \in \mathbb{R}^n \triangleq \mathbb{R}^{n \times 1}.$$

In this case we denote f by T_A .

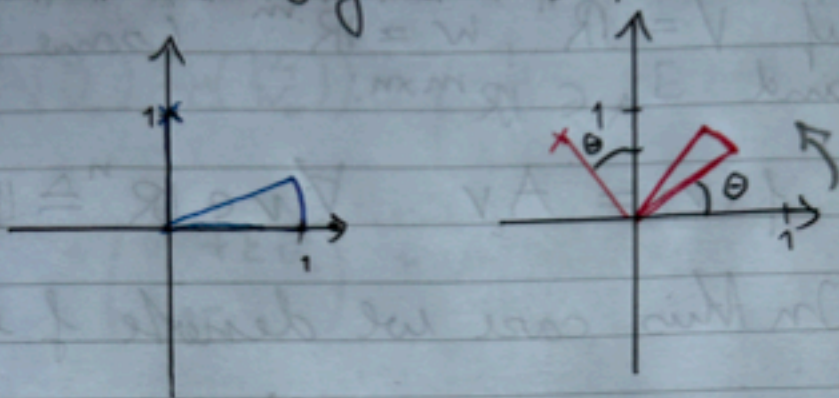
Examples 198: (for $\mathbb{R}^2$)

- (a) f in Motivation 196 is called a shearing in direction of the x-axis with factor k .

You can consider other coordinates (i.e. another basis than B_{st})



(b) rotation by $\theta \in [0, 2\pi[$



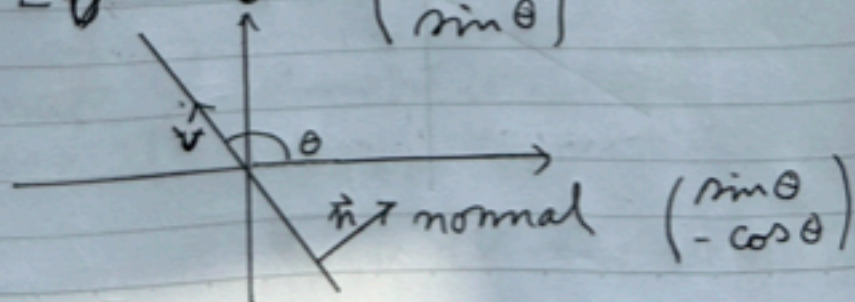
$$T \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} =: R(\theta)$$

E.g. for $\theta = \pi$ we get $T \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

the point reflection at the origin

(c) reflection ^{about} ~~the~~ a line

$$L = L_v \quad v = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$



$$T_{\vec{n}} = \begin{pmatrix} \cos(\theta - 90^\circ) \\ \sin(\theta - 90^\circ) \end{pmatrix} = \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix}$$

$$T \left(\underbrace{\quad}_{?} \right) =: S_L$$

$$\text{In } B := \left\{ \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} \right\} - \text{coordinates}$$

we have

$$[S_L(w)]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [w]_B$$

$$S_L(v) = v, \quad S_L(\vec{n}) = -\vec{n}$$

$$(\text{Notation: } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = [S_L]_B)$$

The matrix for S_L relative to B .

We want $[S_L]_{B_M}$

$$\begin{aligned} \text{We have } P_{B \rightarrow B_M} [S_L]_B [W]_B &= [S_L]_{B_M} \\ &= [S_L(W)]_{B_M} \end{aligned}$$

$$\begin{aligned} \text{i.e. } P_{B \rightarrow B_M} [S_L]_B P_{B_M \rightarrow B} P_{B \rightarrow B_M} [W]_B &= [S_L(W)]_{B_M} \\ &= [S_L(W)]_{B_M} \end{aligned}$$

$$\text{So } [S_L]_{B_M} = P_{B \rightarrow B_M} [S_L]_B P_{B_M \rightarrow B}$$

$$= \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

$$\text{So } S_L = T \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

For $\theta = 0$ we get $S_L = T \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

is the reflection about the x -axis.

(d) What is $T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$?

("orthogonal projection onto the x -axis.")

Homework: Learn about

- contraction / dilation
- extension / compression

Exercise 199: Let $v \in \mathbb{R}^{2 \times 1}$ of norm 1.

Let P_v be the orthogonal projection onto L_v . Find

A such that $P_v = T_A$.

What happens in $\mathbb{R}^3$?

Example 200:

(a) shearing:

$$\begin{pmatrix} 1 & k \\ & 1 \\ & & 1 \end{pmatrix} \text{ and similar}$$

matrices (similar means conjugate)

Above matrix: translation of a point in $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ direction

proportional to its distance

to the $x-z$ plane.

(b) rotation around a line (called "rotation axis")

Ex: around x -axis:

$$T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

In general $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^{3 \times 1}$ $\|v\|_2 = 1$

W.l.o.g. $v_1 \neq 0$.

$R_v(\theta)$ = rotation around L_v by angle θ .

Find A ?

Step 1: find $\{v, v', v''\}$ an orthonormal basis, i.e.

$$\|v\|_2 = \|v'\|_2 = \|v''\|_2 = 1$$

and

$$v \cdot v' = v \cdot v'' = v' \cdot v'' = 0$$

We also want a right handed basis, i.e.

$$\det \begin{pmatrix} [v]_{B_{\text{std}}} & [v']_{B_{\text{std}}} & [v'']_{B_{\text{std}}} \end{pmatrix} > 0.$$

$$\text{Take } v' = \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{v_1^2 + v_2^2}}$$

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$$V'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ -v_2 & v_1 & 0 \end{vmatrix} \frac{1}{\sqrt{v_1^2 + v_2^2}}$$

$$= \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_1^2 + v_2^2 \end{pmatrix} \frac{1}{\sqrt{v_1^2 + v_2^2}}$$

$$P_{B \rightarrow B_N} = \begin{pmatrix} v_1 & -v_2/\sqrt{v_1^2 + v_2^2} & -v_2 v_3/\sqrt{v_1^2 + v_2^2} \\ v_2 & v_1/\sqrt{v_1^2 + v_2^2} & -v_1 v_3/\sqrt{v_1^2 + v_2^2} \\ v_3 & 0 & \sqrt{v_1^2 + v_2^2} \end{pmatrix}$$

$$[R_V(\theta)]_B = \begin{pmatrix} 1 & & \\ & \cos \theta & -\sin \theta \\ & \sin \theta & \cos \theta \end{pmatrix}$$

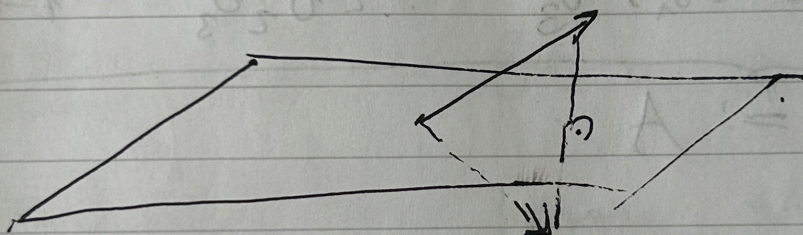
$$[R_V(\theta)]_{B_N} = P_{B \rightarrow B_N} \begin{pmatrix} 1 & & \\ & \cos \theta & -\sin \theta \\ & \sin \theta & \cos \theta \end{pmatrix} P_{B_N \rightarrow B}$$

(c) reflection about a (plane/line/pt)

We only consider plane here.

Ex: $E = x-y$ plane

$$\text{Refl}_E = T \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$



In general: $E = H_v$, $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$
non-zero.

W.l.o.g. $\|v\|_2 = 1$.

Formula: $\text{Refl}_{H_v}(w) = w - 2(w \cdot v)v$

(Think about how to describe the reflection about a line.)

$$[\text{Ref}_{H_v}]_{B_{st}} = \begin{pmatrix} [\text{Ref}_{H_v}(e_1)]_{st} & [\text{Ref}_{H_v}(e_2)]_{st} & [\text{Ref}_{H_v}(e_3)]_{st} \end{pmatrix}$$

$$= \begin{pmatrix} (1-2v_1)v_1 & -2v_2v_1 & -2v_3v_1 \\ (1-2v_1)v_2 & 1-2v_2v_2 & -2v_3v_2 \\ (1-2v_1)v_3 & -2v_2v_3 & 1-2v_3v_3 \end{pmatrix}$$

$=: A$

$$\text{Ref}_{H_v} = T_A$$

Terminology 201: Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

be a linear map.

Then $[f]_{st} = ([f(e_1)]_{st}, \dots, [f(e_n)]_{st})$

is called the standard matrix of f .

Remark 202: If A is the standard matrix of f then $f = T_A$.

Proof: $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$

$$\begin{aligned} f(v) &= \sum v_i f(e_i) = \sum v_i [f(e_i)]_M \\ &= Av \quad \square \end{aligned}$$

Def 203: Let V be a v.s.

A subset of the form $v+W$ with $v \in V$ and $W \leq V$ is called an affine subspace.

If $\dim V < \infty$, then we call $\dim W$ the dimension of $v+W$. But $\dim(v+W) := \dim W$.

Prop 204: Let $f: V \rightarrow W$ be linear.

Then 1) f maps an affine subspace onto an affine subspace, e.g. an affine

line onto a point or
an affine line.

2) The kernel of f -

$$\ker(f) := \{v \in V \mid f(v) = 0_W\}$$

is a subspace of V .

and if $\dim V < \infty$ we have
 $\dim V = \dim \ker(f) + \dim \operatorname{im}(f)$

"Dimension Theorem for
linear maps"

Proof: $v + U$ an affine subspace
of V .

$$f(v + U) = f(v) + f(U)$$

$$f(U) \leq W \quad (\text{subspace criterion})$$

So $f(v + U)$ is a subspace.

Case $L_{v'} = U : f(v + L_{v'}) = f(v) + R f(v')$

a pt. or an affine line.

End of Lecture 29th of Nov 2023

2) Take $B = \{v_1, \dots, v_n\}$ basis of V such that $C = \{f(v_1), \dots, f(v_r)\}$ is a basis of $\text{im}(f) =: U$.

Represent $f: V \rightarrow U$ by a matrix

$$\begin{aligned} A &= [f]_{C,B} \\ &= ([f(v_1)]_C, \dots, [f(v_n)]_C) \\ &= \begin{pmatrix} 1 & \dots & 0 & & \\ & \ddots & & & \\ & & 1 & & \\ & & & & * \end{pmatrix} \end{aligned}$$

$[f]_{C,B}$ is called the matrix for f relative to the bases B and C

Then $\text{null}(A) = \{[v]_B \mid v \in \ker(f)\}$

Thus $\dim \text{null}(A) = \dim \ker(f)$

$$\overset{\parallel}{=} n - r = \dim V - \dim(\text{im}(f))$$

□

Later we will prove: $\sqrt{3} = \sqrt{3}$

Theorem 205:

Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map
s.t. $\|f(v)\| = \|v\|$ for all $v \in \mathbb{R}^3$.

Suppose $\det [f]_{\mathcal{A}} = 1$.

Then f is a rotation.

Example 206:

$$f = T_A \quad \text{for} \quad A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$f(v) = \begin{pmatrix} \frac{v_1}{\sqrt{2}} + \frac{v_2}{\sqrt{2}} \\ -v_3 \\ -\frac{v_1}{\sqrt{2}} + \frac{v_2}{\sqrt{2}} \end{pmatrix} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{aligned} \|f(v)\|^2 &= \frac{1}{2}(v_1 + v_2)^2 + v_3^2 + \frac{1}{2}(v_2 - v_1)^2 \\ &= v_1^2 + v_2^2 + v_3^2 = \|v\|^2 \end{aligned}$$

f is linear

$\Rightarrow f$ is a rotation by Theorem 205.

Find: 1) line of rotation
2) rotation angle.

1) Solve $Av = v$

$$\Leftrightarrow (A - I_3)v = 0$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} - 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & -1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Sol}^n = \{ (\lambda, \lambda(\sqrt{2}-1), \lambda(1-\sqrt{2}))^T \mid \lambda \in \mathbb{R} \}$$

is the line of rotation.

$$v^{(1)} := \begin{pmatrix} 1 \\ \sqrt{2}-1 \\ 1-\sqrt{2} \end{pmatrix}$$

$$v^{(2)} := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \perp v^{(1)}$$

$$f(v^{(2)}) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \cos \theta &= (v^{(2)} \cdot f(v^{(2)})) \frac{1}{\|v^{(2)}\| \|f(v^{(2)})\|} \\ &= \left(\frac{1}{\sqrt{2}} - 1 \right) \frac{1}{2} \\ &= \frac{1 - \sqrt{2}}{2\sqrt{2}} \end{aligned}$$

θ is 98.42° or -98.42°

Which one?

Check right hand ~~rule~~ rule

$$\begin{vmatrix} 1 & \sqrt{2} - 1 & 1 - \sqrt{2} \\ & 1 & 1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 - \sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} - \begin{vmatrix} 1 & \sqrt{2} - 1 \\ \frac{1}{\sqrt{2}} & -1 \end{vmatrix}$$

$\uparrow$
 2^{nd} row

$$= 1 - (-1 - (1 - \frac{1}{\sqrt{2}})) = \frac{1}{\sqrt{2}} + 3 > 0$$

$\Rightarrow \theta$ is 98.42°