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$$A = \begin{pmatrix} 1 & 2 \\ 1 & \end{pmatrix} \quad A^T A = \begin{pmatrix} 1 & 1 \\ 2 & \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\mathrm{Spec}(A^T A) = \{3+\sqrt{5},\ 3-\sqrt{5}\}$$

So $\left\| \begin{pmatrix} 1 & 2 \\ 1 & \end{pmatrix} \right\| = \sqrt{3+\sqrt{5}}$

V.3. Conventional way to diagonaliza matrix (if diagonalizable)

Steps: $A \in \mathbb{R}^{n \times n}$ given and we have given the information that A is diagonalizable/$\mathbb{R}$, e.g. we know

- $m_a(A,\lambda) = m_g(A,\lambda)$ $\forall \lambda \in \mathrm{Spec}(A)$

- and $\mathrm{Spec}(A) = \mathrm{Spec}_{\mathbb{R}}(A)$.

Step 1: Compute $\mathrm{Spec}(A)$.

Step 2: Compute eigenspaces
$\mathrm{Eig}(A, \lambda_i)$

Step 3: Put $P = (v_1, v_2, v_3, \dots, v_n) \in \mathbb{R}^{n\times n}$

were $\{v_1, \dots, v_n\}$ ~~or~~ is an eigenbasis of $\mathbb{R}^n$ for A.

Example:

$$A = \begin{pmatrix} 1 & -1 & 5 & -4 \\ 1 & 3 & -5 & 4 \\ 1 & 1 & -3 & 4 \\ 1 & 1 & -2 & 3 \end{pmatrix}$$

Step 1:

$$P_A(\lambda) = (-1)^4 \begin{vmatrix} 1-\lambda & -1 & 5 & -4 \\ 1 & 3-\lambda & -5 & 4 \\ 1 & 1 & -3-\lambda & 4 \\ 1 & 1 & -2 & 3-\lambda \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 1 & -2 & 3-\lambda \\ 1 & 3-\lambda & -5 & 4 \\ 1 & 1 & -3-\lambda & 4 \\ 1-\lambda & -1 & 5 & -4 \end{vmatrix}$$

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$$= - \begin{vmatrix} 1 & 1 & -2 & 3-\lambda \\ 0 & 2-\lambda & -3 & 1+\lambda \\ 0 & 0 & -1-\lambda & 1+\lambda \\ 0 & -2+\lambda & 7-2\lambda & -\lambda^2+4\lambda-7 \end{vmatrix}$$

$$= - \begin{vmatrix} 2-\lambda & -3 & 1+\lambda \\ & -1-\lambda & 1+\lambda \\ & 4-2\lambda & -\lambda^2+5\lambda-6 \end{vmatrix}$$

$$= (\lambda-2)(1+\lambda) \begin{vmatrix} -1 & 1 \\ 4-2\lambda & -\lambda^2+5\lambda-6 \end{vmatrix}$$

$$= (\lambda-2)(\lambda+1)\left[\lambda^2-3\lambda+2\right]$$

$$= (\lambda-2)^2(\lambda+1)(\lambda-1).$$

$\Rightarrow \operatorname{Spec}(A) = \{-1, +1, 2\}$

Step 2:

$$\operatorname{Eig}(A,-1) = \operatorname{null}\left(\begin{vmatrix} 2 & -1 & 5 & -4 \\ 1 & 4 & -5 & 4 \\ 1 & 1 & -2 & 4 \\ 1 & 1 & -2 & 4 \end{vmatrix}\right)$$

$$= \operatorname{null}\left(\begin{vmatrix} 1 & 1 & -2 & 4 \\ 1 & 4 & -5 & 4 \\ 2 & -1 & 5 & -4 \end{vmatrix}\right) =$$

$$= \operatorname{null}\left(\begin{pmatrix} 1 & 1 & -2 & 4 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 9 & -12 \end{pmatrix}\right) = \operatorname{null}\left(\begin{pmatrix} 1 & 1 & -2 & 4 \\ & 1 & -1 & \\ & -1 & 3 & -4 \end{pmatrix}\right)$$

$$= \operatorname{null}\left(\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -4 \end{pmatrix}\right) = \operatorname{null}\left(\begin{pmatrix} 1 & 0 & 0 & 2 \\ & 1 & 0 & -2 \\ & & 1 & -2 \end{pmatrix}\right)$$

$$= \mathbb{R}\begin{pmatrix} -2 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\operatorname{Eig}(A, 1) = \operatorname{null}\left(\begin{pmatrix} 0 & -1 & 5 & -4 \\ 1 & 2 & -5 & 4 \\ 1 & 1 & -4 & 4 \\ 1 & 1 & -2 & 2 \end{pmatrix}\right)$$

$$= \operatorname{null}\left(\begin{pmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & -1 & 5 & -4 \\ 0 & 0 & -2 & 2 \end{pmatrix}\right)$$

$$= \operatorname{null}\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & +1 & -1 \end{pmatrix}\right) = \operatorname{null}\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 \\ & & 1 & -1 \end{pmatrix}\right)$$

$$= \mathbb{R}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\operatorname{Eig}(A, 2) = ? = \mathbb{R}\begin{pmatrix} 1 \\ 0 \\ +1 \\ 1 \end{pmatrix} + \mathbb{R}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 5 & -4 \\ 1 & 1 & -5 & 4 \\ 1 & 1 & -5 & 4 \\ 1 & 1 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & -1 \\ & & 1 & -1 \\ & & & 0 \\ & & & 0 \end{pmatrix}$$

<u>Step 3:</u> $$P := \begin{pmatrix} -2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad P^{-1} A P = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}$$