App II ~~S~~ Spectral norm of a matrix

What should be a "length" of a matrix and the distance of two matrices?

Let $A \in \mathbb{R}^{m \times n}$.

Put $\|A\| = \sup \left\{ \frac{\|Av\|}{\|v\|} \mid v \in \mathbb{R}^n, v \neq 0 \right\}$

$\in \mathbb{R}^{\geq 0} \cup \{+\infty\}$.

Proposition II 4.1 $\|\cdot\|$ is a norm on $\mathbb{R}^{m \times n}$.

Proof: 1) $\|A\| = \sup \{ \|Av\| \mid \|v\| = 1 \}$

$$= \sup_{\|v\|=1} \left(\sum_{i=1}^{m} \left(\left| \sum_{j=1}^{n} a_{ij} v_j \right| \right)^2 \right)^{\frac{1}{2}}$$

We have $\left| \sum_{j=1}^{n} a_{ij} v_j \right| \leq \sum_{j=1}^{n} |a_{ij}| |v_j|$

$$\leq C \cdot \sum_{j=1}^{n} |v_j| \ , \quad C = \max_{k, j} |a_{kj}|$$

$$\Rightarrow \|A\| \le \sup_{\|v\|=1} \sqrt{m}\, C \left(\sum_{j=1}^{n} |v_j|\right)$$

$$\le \sqrt{m} \cdot C \cdot n \cdot 1 < \infty$$

$$\left(\sum_{j=1}^{n} |v_j| \le n \cdot \|v\| = n \cdot 1. \right)$$

Exercise: Verify for $A, B \in \mathbb{R}^{n \times m}$, $\lambda \in \mathbb{R}$:

(definiteness) $\|A\| = 0 \iff A = 0$

(~~homogeneity~~) $\|\lambda A\| = |\lambda| \|A\|$ (absolute homogenity)

(Δ-inequality.) $\|A + B\| \le \|A\| + \|B\|$

$\square$

<u>Theorem 24.2:</u> $A \in \mathbb{R}^{m \times n}$:

Then $\|A\| = \max\{\sqrt{\lambda} \mid \lambda \in \mathrm{Spec}(A^T A)\}$

That is why $\|A\|$ is called the <u>spectral norm</u> of A.

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Proof: · $A^T A$ is symmetric $\Rightarrow \mathrm{Spec}(A) = \mathrm{Spec}_{\mathbb{R}}(A)$.

· Exercise $\forall \lambda \in \mathrm{Spec}(A): \lambda \geq 0$.

Take an ON eigenbasis for $A^T A$
$v_1, v_2, \dots, v_n$.
say with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Exercise: $\|Av\|^2 \leq \lambda_1 \|v\|^2 \quad \forall v \in \mathbb{R}^n$.
and
$$\|A v_1\|^2 = \lambda_1 \|v_1\|^2$$

Thus $\|A\| = \sqrt{\lambda_1}$ $\square$

Example 243:

$$\left\| \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \right\| = \sup_{\|v\|=1} \left\| \begin{pmatrix} v_1 + 2v_2 \\ v_1 \end{pmatrix} \right\|$$
$$= \sup_{\|v\|=1} \sqrt{2v_1^2 + 4v_2^2 + 4v_1 v_2}$$
$$= \sup_{\|v\|=1} \sqrt{2}\sqrt{1 + v_2^2 + 2v_1 v_2} = ?$$

$A = \begin{pmatrix} 1 & 2 \\ 1 & \end{pmatrix}$ $A^T A = \begin{pmatrix} 1 & 1 \\ 2 & \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$

$\mathrm{Spec}(A^T A) = \{3+\sqrt{5}, 3-\sqrt{5}\}$

So $\left\|\begin{pmatrix} 1 & 2 \\ 1 & \end{pmatrix}\right\| = \sqrt{3+\sqrt{5}}$

V.3. <u>Conventional way</u>
<u>to diagonaliza matrix</u>
(if diagonalizable)

<u>Steps:</u> $A \in \mathbb{R}^{n \times n}$ given
and we have given the information that A is diagonalizable/$\mathbb{R}$, i.e. g. we know
- $m_a(A, \lambda) = m_g(A, \lambda)$
  $\forall_{\lambda} \in \mathrm{Spec}(A)$
- and $\mathrm{Spec}(A) = \mathrm{Spec}_{\mathbb{R}}(A)$.

<u>Step 1:</u> Compute Spec(A).