

App II Spectral norm of a matrix

What should be a "length" of a matrix and the distance of two matrices?

Let $A \in \mathbb{R}^{m \times n}$.

$$\text{Def } \|A\| = \sup \left\{ \frac{\|Av\|}{\|v\|} \mid v \in \mathbb{R}^n, v \neq 0 \right\}$$

$$(v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}) \in \mathbb{R}^{\geq 0} \cup \{+\infty\}$$

Proposition 2.41: $\|\cdot\|$ is a norm on $\mathbb{R}^{m \times n}$.

Proof: 1) $\|A\| = \sup \left\{ \|Av\| \mid \|v\| = 1 \right\}$

$$= \sup_{\|v\|=1} \left(\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} v_j \right)^2 \right)^{\frac{1}{2}}$$

We have $\left| \sum_{j=1}^n a_{ij} v_j \right| \leq \sum_{j=1}^n |a_{ij}| |v_j|$

$$\leq C \cdot \sum_{j=1}^n |v_j|, \quad C = \max_{k,j} |a_{kj}|$$

$$\Rightarrow \|A\| \leq \sup_{\|v\|=1} \sqrt{m} \cdot C \left(\sum_{j=1}^n |v_j| \right)$$

$$\leq \sqrt{m} \cdot C \cdot n \cdot 1 < \infty$$

$$\uparrow$$

$$\left(\sum_{j=1}^n |v_j| \leq n \cdot \|v\| = n \cdot 1 \right)$$

Exercise: Verify for $A, B \in \mathbb{R}^{n \times m}$, $\lambda \in \mathbb{R}$:

(definition) $\|A\| = 0 \Leftrightarrow A = 0$

$$\|\lambda A\| = |\lambda| \|A\| \quad (\text{absolute homogeneity})$$

$$(\triangle \text{-inequality}) \quad \|A + B\| \leq \|A\| + \|B\|$$

□

Theorem 242: $A \in \mathbb{R}^{m \times n}$:

$$\text{Then } \|A\| = \max \{ \sqrt{\lambda} \mid \lambda \in \text{Spec}(A^T A) \}$$

That is why $\|A\|$ is called
the spectral norm of A .

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Proof: $A^T A$ is symmetric $\Rightarrow \text{Spec}(B)$
 $\text{Spec}_{\mathbb{R}}(B)$.

Exercise $\forall \lambda \in \text{Spec}(B): \lambda \geq 0$.

Take an ON eigenbasis for $A^T A$

$v_1, v_2, \dots, v_n$.

They with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Exercise: $\|Av\|^2 \leq \lambda_1 \|v\|^2 \quad \forall v \in \mathbb{R}^n$

and

$$\|Av_1\|^2 = \lambda_1 \|v_1\|^2$$

Thus $\|A\| = \sqrt{\lambda_1}$ $\square$

Example 243:

$$\left\| \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \right\| = \sup_{\|v\|=1} \left\| \begin{pmatrix} v_1 + 2v_2 \\ v_1 \end{pmatrix} \right\|$$

$$= \sup_{\|v\|=1} \sqrt{2v_1^2 + 4v_2^2 + 4v_1v_2}$$

$$= \sup_{\|v\|=1} \sqrt{2} \sqrt{1 + v_2^2 + 2v_1v_2} = ?$$

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$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad A^T A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\ = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\text{Spec}(A^T A) = \{3 + \sqrt{5}, 3 - \sqrt{5}\}$$

$$\text{So } \|(1 \ 2)\| = \sqrt{3 + \sqrt{5}}$$

7.3. Conventional way
to diagonalize a matrix
(if diagonalizable)

Steps: $A \in \mathbb{R}^{n \times n}$ given
and we have given the
information that A is
diagonalizable/ $\mathbb{R}$, i.e. we know

- $m_A(A, \lambda) = m_{\mathbb{R}}(A, \lambda)$
- $\forall \lambda \in \text{Spec}(A)$

• and $\text{Spec}(A) = \text{Spec}_{\mathbb{R}}(A)$.

Step 1: Compute $\text{Spec}(A)$.