is an eigen basis for A and also an ON basis.

V.2. Applications.

App I Linear differential equations

$$(*) \quad y' = Ay \qquad A \in \mathbb{R}^{n \times n} \text{ fixed.}$$

Find $y \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{R}^n)$
(infinitely often differentiable map)
which solves $(*)$.

Fact: For every $v \in \mathbb{R}^n$ there exists exactly one solution y such that $y(0) = v$.

How to find this y?
We only consider the case where A is diagonalizable over $\mathbb{R}$.

$\Rightarrow \exists\, P \in \mathbb{R}^{n \times n}$ invertible such that

$$P^{-1} A P = D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

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The columns of P form an A-eigenbasis of $\mathbb{R}^n$, say $\{v_1, \dots, v_n\}$

Given: $v \in \mathbb{R}^n$: $v = \sum_{i=1}^{n} \mu_i v_i$.

Put $y(t) := \sum_{i=1}^{n} \mu_i v_i e^{\lambda_i t}$

Then $y'(t) = \sum_{i=1}^{n} \mu_i \lambda_i v_i e^{\lambda_i t}$

and
$$A(y(t)) = \sum_{i=1}^{n} \mu_i A v_i e^{\lambda_i t}$$
$$= \sum_{i=1}^{n} \mu_i \lambda_i v_i e^{\lambda_i t}$$
$$= y'(t) \checkmark$$

Example 240: $y' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$p_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 3-\lambda & 1 \\ 3 & 1 & 2-\lambda \end{vmatrix}$$

$$= \lambda^3 - (\cancel{\ }+6)\lambda^2 + (6+2+3-1-9-4)\lambda$$
$$-(-18)$$

$$= \lambda^3 - 6\lambda^2 - 3\lambda + 18$$

$$= (\lambda-6)(\lambda^2-3) = (\lambda-6)(\lambda-\sqrt{3})(\lambda+\sqrt{3}).$$

$$\mathrm{Eig}(A,6) = \mathbb{R}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\mathrm{Eig}(A,\sqrt{3}) = \mathrm{null}\left(\begin{pmatrix}1-\sqrt{3} & 2 & 3\\ 2 & 3-\sqrt{3} & 1\\ 3 & 1 & 2-\sqrt{3}\end{pmatrix}\right)$$

$$= \mathrm{null}\left(\begin{pmatrix}-5-\sqrt{3} & -7+3\sqrt{3} & 0\\ 2 & 3-\sqrt{3} & 1\\ -1+2\sqrt{3} & -8+5\sqrt{3} & 0\end{pmatrix}\right)$$

$$= \mathbb{R}\begin{pmatrix}7-3\sqrt{3}\\ -5-\sqrt{3}\\ -2+4\sqrt{3}\end{pmatrix}$$

$$\mathrm{Eig}(A,-\sqrt{3}) = \mathrm{null}\left(\begin{pmatrix}-5+\sqrt{3} & -7-3\sqrt{3} & 0\\ 2 & 3+\sqrt{3} & 1\\ 0 & 0 & 0\end{pmatrix}\right)$$

$$= \mathbb{R}\begin{pmatrix}7+3\sqrt{3}\\ -5+\sqrt{3}\\ -2-4\sqrt{3}\end{pmatrix}.$$

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So the general sol^n of $y' = Ay$ is

$$y(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t\cdot 6} + c_2 \begin{pmatrix} 7-3\sqrt{3} \\ -5-\sqrt{3} \\ -2+4\sqrt{3} \end{pmatrix} e^{\sqrt{3}t}$$

$$+ c_3 \begin{pmatrix} 7+3\sqrt{3} \\ -5+\sqrt{3} \\ -2-4\sqrt{3} \end{pmatrix} e^{-\sqrt{3}t}.$$

$c_1, c_2, c_3 \in \mathbb{R}$.

Initial condition $y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 7-3\sqrt{3} \\ -5-\sqrt{3} \\ -2+4\sqrt{3} \end{pmatrix}$$

$$+ c_3 \begin{pmatrix} 7+3\sqrt{3} \\ -5+\sqrt{3} \\ -2-4\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad c_1 = 1/3, \; c_2 = \frac{-3+5\sqrt{3}}{132\cdot\sqrt{3}}, \; c_3 = \frac{3+5\sqrt{3}}{132\sqrt{3}}$$