LINEAR ALGEBRA 1
PROBLEM SHEET 11

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Problem 1 (25, diagonalization)**.** Which of the following matrices A are diagonalizable over $\mathbb{R}$? If so, then compute a transition matrix P such that $\mathrm{P}^{-1}\mathrm{AP}$ is diagonal.

(i)
$$\begin{pmatrix} -2 & 5 & -4 \\ 4 & -3 & 4 \\ 3 & -5 & 5 \end{pmatrix}$$

(ii)
$$\begin{pmatrix} -1 & 2 & 2 & 0 \\ -4 & 5 & 2 & 0 \\ 8 & -4 & -1 & 0 \\ -4 & 2 & 2 & 3 \end{pmatrix}$$

(iii)
$$\begin{pmatrix} -2 & 1 & -1 & 1 \\ -4 & 1 & -4 & 0 \\ 1 & -1 & 0 & -1 \\ 2 & 0 & 2 & 1 \end{pmatrix}$$

Problem 2 (20, finite dimensional inner product space)**.** Let A be a real symmetric matrix of size n. Prove that the following assertions are equivalent:

(i) $\langle , \rangle_{\mathrm{A}}$ is an inner product.
(ii) All eigenvalues of A are positive.

Problem 3 (10+5+10*, infinite dimensional inner product space)**.** We consider on $\mathrm{C}([0,1])$, the set of continuous real-valued functions on $[0,1]$, the pairing

$$\langle f, g\rangle := \int_0^1 f(x)g(x)dx.$$

(i) Prove that $\langle , \rangle$ is an inner product.
(ii) We restrict $\langle , \rangle$ on the subset V of all real polynomial functions on $[0,1]$. Given a non-negative integer n, let p_n be the polynomial function given by $p_n(x) = x^n$, $x \in [0,1]$. Find
   (a) the orthogonal complement of p_1 in V.
   (b) the orthogonal complement of $\{p_1, p_3, p_5, \ldots\}$ in V.

Problem 4 (10+10+10*, finite dimensional inner products spaces)**.** Consider the following pairings $\langle , \rangle_{\mathrm{A}}$. Determine if the pairing is an inner product and in any case compute the Gram matrix with respect to the given basis.

(i)
$$\begin{pmatrix} 5 & .1 \\ 1 & 2 \end{pmatrix}, \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

(ii)
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$

Date: Please hand in before the lecture by **27th of December 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.

(iii)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \ \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$