LINEAR ALGEBRA 1
PROBLEM SHEET 10

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Problem 1 (40, complex numbers). (i) Prove that $(\mathbb{C}, +, \cdot_{\mathbb{C}})$ satisfies the distributivity law:

$$(a,b) \cdot_{\mathbb{C}} ((c,d) + (e,f)) = ((a,b) \cdot_{\mathbb{C}} (c,d)) + ((a,b) \cdot_{\mathbb{C}} (e,f))$$

(ii) Let V be a real vector space and $f : \mathrm{V} \to \mathrm{V}$ be an $\mathbb{R}$-linear map such that $f \circ f = -\mathrm{id}_{\mathrm{V}}$. ($f(f(v)) = -v$) We define

$$(a + bi) \odot v := av + bf(v).$$

Prove that $(\mathrm{V}, +, \odot)$ is a complex vector space.

(iii) $\mathrm{V} = \mathbb{R}^2$. We define a complex scalar multiplication $\odot$ via

$$(a + bi) \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} ax_1 + b(x_1 - 2x_2) \\ ax_2 + b(x_1 - x_2) \end{pmatrix}$$

Prove that $(\mathrm{V}, +, \odot)$ is a complex vector space.

(iv) $\mathbb{C} = \mathbb{R}^2$ is a complex vector space using complex multiplication as scalar multiplication. Find the corresponding $\mathbb{R}$-linear map f from (ii) which describes this complex scalar multiplication.

Problem 2 (30, multiplicities). Find the geometric and algebraic multiplicities for the following matrices A for all $\lambda \in \mathrm{Spec}(\mathrm{A})$.

(i)

$$\begin{pmatrix} 1 & 1 & & & & & & & \\ & 1 & 1 & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & 1 & & & & \\ & & & & 1 & & & & \\ & & & & & 2 & & & \\ & & & & & & 2 & & \\ & & & & & & & 3 & 1 \\ & & & & & & & & 3 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

(iii)

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ -1 & 8 & 8 & 2 \\ 1 & -4 & -4 & -1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Problem 3 (10, similarity over $\mathbb{C}$ and $\mathbb{R}$). Let A and B be real square matrices of size n. Suppose A and B are similar over $\mathbb{C}$, i.e. there exists an invertible matrix $\mathrm{C} \in \mathbb{C}^{n\times n}$ such that $\mathrm{C}^{-1}\mathrm{AC} = \mathrm{B}$. Prove that A and B are similar over $\mathbb{R}$, i.e. there exists an invertible matrix $\mathrm{F} \in \mathbb{R}^{n\times n}$ such that $\mathrm{F}^{-1}\mathrm{AF} = \mathrm{B}$. *Hint: Consider for a certain matrix* $\mathrm{C} = \mathrm{C}_1 + i\mathrm{C}_2,\ \mathrm{C}_1, \mathrm{C}_2 \in \mathbb{R}^{n\times n}$, *the polynomial function* $p(\lambda) = \det(\mathrm{C}_1 + \lambda\mathrm{C}_2)$, $\lambda \in \mathbb{C}$.

Date: Please hand in before the lecture by **20th of December 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.

Problem 4 (40, geometric multiplicity 1 case)**.** Let $a_0, a_1, a_2, \ldots, a_{n-1}$ be real numbers.

(i) Compute the characteristic polynomial of

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & \ddots & & \vdots & \vdots \\ & \ddots & \ddots & \vdots & \vdots \\ & & 1 & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}.$$

(ii) Let λ be a complex eigenvalue of A. Show that its geometric multiplicity is 1.

(iii) Is the matrix

$$\begin{pmatrix} & & & 2 \\ 1 & & & 3 \\ & 1 & & -1 \\ & & 1 & -3 \end{pmatrix}$$

diagonlizable?

(iv) Find a real matrix A of size 4 with complex eigenvalues $1-i, 1+i, 2, 3$. Given two such matrices, are they similar over $\mathbb{R}$?

Problem 5. aufgabe[20*, complex and real eigenspaces] Let A be a real matrix of size n and $\lambda \in \mathrm{Spec}_{\mathbb{R}}(\mathrm{A})$. Then:

(i) $\mathrm{Eig}_{\mathbb{C}}(\mathrm{A}, \lambda) = \mathrm{Eig}_{\mathbb{R}}(\mathrm{A}, \lambda) + i\mathrm{Eig}_{\mathbb{R}}(\mathrm{A}, \lambda)$.

(ii) $\dim_{\mathbb{C}} \mathrm{Eig}_{\mathbb{C}}(\mathrm{A}, \lambda) = \dim_{\mathbb{R}} \mathrm{Eig}_{\mathbb{R}}(\mathrm{A}, \lambda)$.