1) a) (A) no (B) no (C) yes (D) no

~~3~~ b) (A) T (B) F (C) T (D) T

c) (A) ✓ (B) ✓ (C) no (D) no

2) a) $(\operatorname{adj}(A))^{-1} = \frac{1}{\det(A)} A = \underline{\underline{\frac{1}{2}\begin{pmatrix} 2 & 2 \\ 3 & 4 \end{pmatrix}}}$

b) $P_{B' \to B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow P_{B \to B'} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \underline{\underline{\frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix}}}$

c) $\det(A) = \underline{\underline{0}}$ because $rk(A) = 1$ and $n \geq 2$

3) $A^2 - AB = I_3 = A(A-B)$

$\Rightarrow A - B = A^{-1} = \frac{1}{(-1)} \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow B = -\begin{pmatrix} 1 & 1 & \\ & -1 & 1 \\ & & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ & -1 & 1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

<u>Check:</u> $A(A-B) = \begin{pmatrix} 1 & 1 & -1 \\ & -1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \\ & -1 & 1 \\ & & 1 \end{pmatrix} = I_3$ ✓

4) $A = \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & 3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix}$

$\Rightarrow col(A) = span\left\{ \overset{v_1}{\begin{pmatrix} 4 \\ 6 \\ 3 \\ 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} -1 \\ 0 \\ 6 \\ 9 \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 2 \\ 3 \\ -3 \\ 0 \end{pmatrix}} \right\}$, because $c_1(A)$ and $c_2(A)$ are linear combin. of $c_3(A)$ and $c_4(A)$.

- $\{v_1, v_2, v_3\}$ <u>is a basis</u>, because $\begin{vmatrix} 6 & 0 & 3 \\ 3 & 6 & -3 \\ 0 & 9 & 0 \end{vmatrix} = \begin{vmatrix} 9 & 6 & 0 \\ 3 & 6 & -3 \\ 0 & 9 & 0 \end{vmatrix} \neq 0$.
- $\operatorname{rank}(A) = 3$
- $\operatorname{nullity}(A) = 5 - 3 = 2$
- $\begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & 3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & -9 & 9 & -9 \\ 0 & 0 & -12 & 12 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \{(1,3,4,-1,2), (0,0,1,-1,\frac{1}{2}), (0,0,0,0,-\frac{9}{2})\}$ is a basis of $row(A)$

- $\text{null}(A) = \text{span}\underbrace{\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}}_{\text{basis}}$

b) $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \begin{pmatrix} 2 \\ 1 \\ 7 \\ -12 \end{pmatrix} \Rightarrow \lambda_2 = -\frac{4}{3}$

and $3(\lambda_1 - \lambda_3) + (-8) = 7$

and $6\lambda_1 + 3\lambda_3 = 1$

and $4\lambda_1 + 2\lambda_3 = 2 + \frac{4}{3} = \frac{2}{3}$

$\Leftrightarrow \quad 9\lambda_1 = 16 \quad \wedge \quad \lambda_2 = -\frac{4}{3} \quad \wedge \quad \lambda_3 = \frac{1 - 2\lambda_1}{3}$

$\lambda_1 = \frac{16}{9}$

$= \frac{3 - 32}{9}$

$= \frac{-29}{9}$

$\wedge \quad \underbrace{4\lambda_1 + 2\lambda_3}_{\frac{1}{9}(64 - 58) = \frac{6}{9}} = \frac{2}{3}$ ✓

So yes ✓!

Check: $\frac{16}{9}\begin{pmatrix} 4 \\ 6 \\ 3 \\ 0 \end{pmatrix} - \frac{4}{3}\begin{pmatrix} -1 \\ 0 \\ 6 \\ 9 \end{pmatrix} - \frac{29}{9}\begin{pmatrix} 2 \\ 3 \\ -3 \\ 0 \end{pmatrix}$

$= \frac{1}{9}\begin{pmatrix} 64 + 12 - 58 \\ 96 - 87 \\ 48 - 72 + 87 \\ -108 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 18 \\ 9 \\ 63 \\ -108 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 7 \\ -12 \end{pmatrix}$ ✓

c) $\begin{pmatrix} 2 & 3 & 1 & 0 \\ -1 & 0 & 6 & 9 \\ 2 & 3 & -3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -6 & -9 \\ 0 & 3 & 13 & 18 \\ 0 & 3 & 9 & 18 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -6 & -9 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 4 & 0 \end{pmatrix}$

$\Rightarrow \text{col}(A)^\perp = \text{span}\left\{ \begin{pmatrix} -9 \\ -6 \\ 0 \\ 1 \end{pmatrix} \right\}$

5) a) • $0 \in U \cap W$

the structure of the matrices of U and W is closed under $+$ and ~~spat~~ scalar mult. $\Rightarrow$ $U, W \leq \mathbb{R}^{2\times 2}$.

• $U = \text{span}\{\underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{lin. indep.}}\}$ $\Rightarrow$ basis and dim $= 3$

• $W = \text{span}\{\underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{basis}}\}$. dim $W = 3$.

b) $U + W = \mathbb{R}^{2\times 2}$, because $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in U + W$ and $U \subseteq U + W$.

$\Rightarrow \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ basis of $U + W$, dim $= 4$

$\dim(U \cap W) = 3 + 3 - 4 = 2$

$\Rightarrow U \cap W = \text{span}\{\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}}_{\text{basis}}\}$; dim $= 2$ as seen before.

7) a) $3v_1 - 2v_2 = v_3 \Rightarrow \dim(\operatorname{span}\{v_1, v_2, v_3\}) \leq 2$

$\Rightarrow$ There are no 3 linear indep. vectors in this span.

Answer: No

~~8)~~ b) Yes: $\operatorname{rk}(A) = 1 \Rightarrow \exists$ column $\underset{\neq 0}{u}$ of A and $\operatorname{col}(A) = \operatorname{span}\{u\}$.

$\Rightarrow A = uv^T$ for some $v \in \mathbb{R}^{n\times 1} \setminus \left\{\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\right\}$.

8) Yes: $\left(\sum_{i=1}^{3} \lambda_i v_i\right) + \mu_1 w_1 + \mu_2 w_2 = 0$

$\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = -\mu_1 w_1 - \mu_2 w_2$

Multiply with $-\mu_1 w_1 - \mu_2 w_2$ ($\bullet$-prod.)

$\Rightarrow \|\mu_1 w_1 + \mu_2 w_2\|^2 = 0$

$\underset{\text{l. indep.}}{\Rightarrow} \mu_1 = \mu_2 = 0.$

Analogously $\lambda_1 = \lambda_2 = \lambda_3 = 0$. $\square$