LINEAR ALGEBRA 1
PROBLEM SHEET 9

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Problem 1 (10 points, similar matrices)**.** Consider the matrix A

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Prove that A is similar to its transpose.

Problem 2 (10*+10 points, Markov chain)**.** Let R_1 and R_2 be two wildlife reserves for tigers. In one year a tiger in R_1 migrates to R_2 with probability 0.3 and stays within R_1 with probability 0.7. A tiger in R_2 migrates to R_1 with probability 0.2 and stays within R_2 with probability 0.8. At time 0 both reserves have the same number of tigers. $x(0) = (0.5, 0.5)^T$.

(i) Let $x(t) = (x_1(t), x_2(t))^T$ be the distribution of tigers after t years. Prove that the sequence of state vectors $(x(t))_{t\in\mathbb{N}}$ converges to a vector in $\mathbb{R}^2$. Hint: Show that the powers of the transition matrix of the problem converge in finding its eigenvalues. .

(ii) Compute the limit distribution.

Problem 3 (20+10 points, eigenvalues and eigenvectors)**.** (i) Find the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

(ii) Consider the sequence

$$a_{-1} := a_0 := 1,\ a_1 := 3,\ a_{n+1} := 2a_n + a_{n-1},\ n \geqslant 1.$$

Find an explicit formula for a_n and prove your formula. Hint: Consider the matrix

$$\begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix}$$

Problem 4 (10+10, direct sum of eigenspaces)**.** Let V be a vector space and $W_1, W_2, \ldots, W_r$ be subspaces. We call the sum

$$W_1 + W_2 + W_3 + \ldots + W_r$$

a *direct sum* if the zero vector **cannot** be obtained as a sum $w_1 + w_2 + \ldots + w_r$ with $w_i \in W_i$ with some summand being non-zero. Let A be a square matrix of size n and $\lambda_1, \ldots, \lambda_r$ be distinct real eigenvalues of A.

(i) Prove that

$$\mathrm{Eig}(A, \lambda_1) + \mathrm{Eig}(A, \lambda_1) + \ldots + \mathrm{Eig}(A, \lambda_r)$$

is direct.

(ii) We call $\mathrm{Eig}^i(A, \lambda) := \mathrm{null}((A - \lambda I_n)^i)$ the ith *higher eigenspace* of A corresponding to λ. Given any tuple of positive integers $\nu_1, \nu_2, \ldots, \nu_r$, prove that

$$\mathrm{Eig}^{\nu_1}(A, \lambda_1) + \mathrm{Eig}^{\nu_2}(A, \lambda_1) + \ldots + \mathrm{Eig}^{\nu_r}(A, \lambda_r)$$

is direct.

Date: Please hand in before the lecture by **13th of December 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.