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Answers to the review problems

1. a) (A) Yes (B) No (C) No (D) Yes

Proof: For (A) and (D) use the subspace criterion. We only show (A), (D) is proven similarly

For (A) $W := \{ \underline{x} \in \mathbb{R}^3 \mid \underline{x} \cdot (1,5,3) = 0 = \underline{x} \cdot (0,1,-2) \}$

. $\underline{0} \in W$

. $\underline{x}, \underline{y} \in W \Rightarrow (\underline{x}+\underline{y}) \cdot (1,5,3) = (x_1+y_1) + 5(x_2+y_2) + 3(x_3+y_3)$

$= \underline{x} \cdot (1,5,3) + \underline{y} \cdot (1,5,3) \underset{\underline{x},\underline{y} \in W.}{\overset{\uparrow}{=}} 0 + 0 = 0$

analogously $(\underline{x}+\underline{y}) \cdot (0,1,-2) = 0$. Thus $\underline{x}+\underline{y} \in W$

. $\lambda \in \mathbb{R}, \underline{x} \in W \Rightarrow (\lambda x_1, \lambda x_2, \lambda x_3) \cdot (1,5,3)$

$= \lambda x_1 + 3\lambda x_3 + 5\lambda x_2 = \lambda(\underline{x} \cdot (1,5,3)) = \lambda \cdot 0 = 0$

~~Thus~~ $(\lambda \underline{x}) \cdot (0,1,-2) = 0$ similarly

Thus $\lambda \underline{x} \in W$.

So W is a subspace.

For (B) and (C): $\underline{0}$ is not an element of those sub. □

b) (A) F (B) F (C) True (D) True

Proof: (A) $\det\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0.$

(B) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(C) and (D) $\det(A - B) \neq 0$, so $A - B$ is invertible and has full rank. □

c) (A), (B), (C) are possible. (D) not.

Proof: Take a basis $\{v_1, \dots, v_6\}$ of V.

(A) $W = \operatorname{span}\{v_1, \dots, v_4\}$, $U := W$

(B) — " — $U := \operatorname{span}\{v_2, v_3, v_4, v_5\}$

(C) — " — , $U := \operatorname{span}\{v_3, v_4, v_5, v_6\}$

(D) $\dim(W \cap U) = \dim W + \dim U - \dim(W+U)$
$\geq 4 + 4 - 6 = 2.$ $\square$

2. a) $a = -2$

<u>Pf:</u> $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2a \\ a & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2a \\ 0 & -a & 1-a \end{vmatrix} = \begin{vmatrix} 1 & -2a \\ 0 & 1-a-2a^2 \end{vmatrix} = 1 - a - 2a^2 \overset{!}{=} 0$

$\Leftrightarrow a \in \{-\frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{1}{2}}\} = \{\frac{1}{2}, -1\}$

Suppose $\begin{pmatrix} -2 \\ -7 \\ -12 \end{pmatrix} = 3a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-7-3a) \begin{pmatrix} 0 \\ 1 \\ -2a \end{pmatrix} + a \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} -2 \\ -7 \\ -12 \end{pmatrix} = \begin{pmatrix} a^2 + 3a \\ -7 \\ 18a + 6a^2 \end{pmatrix} \Leftrightarrow a^2 + 3a + 2 = 0$

$\Leftrightarrow a \in \{-\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{8}{4}}\} = \{-2, -1\}$

$\Leftrightarrow a = -2.$ $\square$

$\uparrow$
basis property
of $a \neq -1$

b) $\det(-2\,\mathrm{adj}(A)) = -72$

<u>Proof:</u> $\mathrm{adj}(A)A = \det(A) I_3$

So $\det(\mathrm{adj}(A)) \cdot (-3) = \det(\det(A) I_3) = (-3)^3$

$\Rightarrow \det(\mathrm{adj}(A)) = 9$ and thus $\det(-2\,\mathrm{adj}(A)) = (-8) \cdot 9$
$= -72.$ $\square$

c) $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 1 & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \\ & & & & 0 \end{pmatrix}$, $A^{n-1} = \begin{pmatrix} & & 1 \\ & 0 & \\ & & \end{pmatrix}$

$\mathrm{rk}(A^{n-1}) = 1.$

3) $A(X-B)=C \Leftrightarrow AX = AB + C = \begin{pmatrix} -11 & 3 \\ 0 & 29 \\ -5 & 6 \end{pmatrix}$

$$\left(\begin{array}{ccc|cc} 1 & -2 & 1 & -11 & 3 \\ 2 & 1 & 3 & 0 & 29 \\ 1 & -1 & 1 & -5 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & -2 & 1 & -11 & 3 \\ 0 & 5 & 1 & 22 & 23 \\ 0 & 1 & 0 & -5 & 6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|cc} 1 & -2 & 1 & -11 & 3 \\ 0 & 5 & 1 & 22 & 23 \\ 0 & 1 & 0 & 6 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|cc} 1 & -2 & 1 & -11 & 3 \\ 0 & 0 & 1 & -8 & 8 \\ 0 & 1 & 0 & 6 & 3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & -8 & 8 \\ 0 & 1 & 0 & 6 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 9 & 1 \\ 0 & 0 & 1 & -8 & 8 \\ 0 & 1 & 0 & 6 & 3 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 9 & 1 \\ 6 & 3 \\ -8 & 8 \end{matrix}}_{X} \\ 0 & 0 & 1 & \end{array}\right)$$

<u>Check</u>: $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{pmatrix} (X - B) = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 15 & -7 \\ 2 & -2 \\ -10 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}$

So $\underline{\underline{X = \begin{pmatrix} 9 & 1 \\ 6 & 3 \\ -8 & 8 \end{pmatrix}}}$.

4) $A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} \Rightarrow \underbrace{\mathrm{adj}(A)\, A}_{\det(A) I_n} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \mathrm{adj}(A) \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$

$$\underset{|A|=1}{\Longrightarrow} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \mathrm{adj}(A) \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix} \underset{1^{st}\text{ row.}}{\Longrightarrow} 1 = C_{11} + 2C_{21} + \dots + nC_{n1}.$$

5) a) $\begin{pmatrix} 1 & -1 & -6 & 2 & 2 \\ 1 & 6 & 1 & -1 & 1 \\ -1 & -4 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -6 & 2 & 2 \\ 0 & 7 & 7 & -3 & -1 \\ 0 & -5 & -5 & 3 & 3 \\ 0 & 2 & 2 & 0 & 2 \end{pmatrix}$

$$\xrightarrow[\mathrm{II}+\mathrm{III}]{} \begin{pmatrix} 1 & -1 & -6 & 2 & 2 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & -5 & -5 & 3 & 3 \\ 0 & 2 & 2 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -6 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\left\{ \begin{pmatrix}1\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\6\\-4\\2\end{pmatrix}, \begin{pmatrix}2\\-1\\1\\0\end{pmatrix} \right\}$ is a basis of W
and $\dim W = 3$

b) $\begin{pmatrix} c_1(A)^T \\ c_2(A)^T \\ c_3(A)^T \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 6 & -4 & 2 \\ 2 & -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 7 & -5 & 2 \\ 0 & -3 & 3 & 0 \end{pmatrix}$

$\Longrightarrow$ So $\begin{pmatrix}0\\ \vdots \\ -1\end{pmatrix} \in U := W^{\perp}$

$\dim U = \dim(\mathbb{R}^4) - \dim W = 4 - 3 = 1.$

So $\left\{ \begin{pmatrix}0\\ \vdots \\ -1\end{pmatrix} \right\}$ is a basis of U.

6) a) $B := \{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$,
because linearly indep. and has 3 elements.

For S: $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$

For S': $\begin{vmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{vmatrix} \underset{\substack{\uparrow \\ 2^{nd} \text{ row}}}{=} -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0.$

So S and S' are linearly indep

$\Rightarrow$ S and S' are a basis of $\mathbb{R}^3$.

$|\ | = 3$

c) $P_{S' \to S} = P_{B \to S} \circ P_{S' \to B} = (P_{S \to B})^{-1} P_{S' \to B}$.

$(P_{S\to B} \mid P_{S' \to B}) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{array}\right)$

$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$

<u>Check:</u> $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ ✓

So $\underline{\underline{P_{S' \to S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}}}.$

c) $P_{S' \to S}[u]_{S'} = [u]_S = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right) \Rightarrow [u]_{S'} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

<u>Check:</u> $P_{S' \to B}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$

$$P_{S \to B}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \checkmark$$

7) It is meant to be projections onto the lines $L_{v_1}, L_{v_2}, L_{v_3}$.

So
$$3 = v_1 \cdot u = v_1 \cdot w = 14k_1$$
$$-3 = v_2 \cdot u = v_2 \cdot w = 14k_2$$
$$-3 = v_3 \cdot u = v_3 \cdot w = 14k_3$$

$$\Rightarrow w = \frac{3}{14}v_1 - \frac{3}{14}v_2 - \frac{3}{14}v_3.$$

v_1, v_2, v_3 are pairwise orthogonal $\Rightarrow \|w\|^2 = \left(\frac{3}{14}\right)^2(\|v_1\|^2 + \cdots + \|v_3\|^2)$

$= \left(\frac{3}{14}\right)^2 \cdot 3(14) \Rightarrow \|w\| = \underline{\underline{\frac{3}{14}\sqrt{42}}}$

<u>Check:</u> $w = (4,1,5,0)\frac{3}{14}$

$\Rightarrow \|w\| = \frac{3}{14}\sqrt{42}$ ✓

δ) . rank$(A) = 1$ because $col(A) = span\{u\}$

. rank$(B) = 2$.

<u>Proof:</u> $B \in \mathbb{R}^{2\times 2}$ $w \in \mathbb{R}^{2\times 1}$

If $Bw = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow w^T Bw = w^T\begin{pmatrix}0\\0\end{pmatrix} = 0$

$\Rightarrow \|Cw\|^2 = 0 \Rightarrow Cw = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

$\uparrow$

$C = (u, v) \in \mathbb{R}^{3\times 2}$

u, v are linearly indep. $\Rightarrow w = \begin{pmatrix}0\\0\end{pmatrix}$

So rank$(B) = 2$ $\square$