Corollary 116: Let $f \in C_b(\mathbb{R}_+, |\cdot|)_+$

$$C^{(N)} = f(S_N^{(N)})$$

$$\Rightarrow \pi(H^{(N)}) = E_{P_N^*}\left[\frac{f(S_N^{(N)})}{(1+r_N)^N}\right]$$

$$\xrightarrow{N \to \infty} E_{P^*}[f(S_T)]\, e^{-rT}$$

where $E_{P^*}[S_T] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(S_0 e^{\sigma\sqrt{T} y + T(r - \frac{\sigma^2}{2})}\right)$

$$\cdot e^{-\frac{y^2}{2}}\, dy.$$

Example 117: Consider a put option:

$$(K - x)^+ \in C_b(\mathbb{R}_+, |\cdot|)_+$$

$$E_{P_N^*}\left[\frac{(K - S_N^{(N)})^+}{(1+r_N)^N}\right] \xrightarrow{N \to \infty} e^{-rT} E_{P^*}[(K - S_T)^+]$$

Put – Call – parity $\Rightarrow$

$$E_{P_N^*}\left[\frac{(S_N^{(N)} - K)^+}{(1+r_N)^N}\right] \xrightarrow{N \to \infty} e^{-rT} E_{P^*}[(S_T - K)^+]$$

$$= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(\dots)^+ e^{-\frac{y^2}{2}}\,dy$$

$(\quad)^+$ is positive for $y > -\dfrac{\left(\ln\frac{S_0}{K} + (r-\frac{\sigma^2}{2})T\right)}{\sigma\sqrt{T}}$

$$d_-(T,S_0) := \frac{\ln\frac{S_0}{K} + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_+(T,S_0) = d_-(T,S_0) + \sqrt{T}\,\sigma$$
$$= \frac{\ln\frac{S_0}{K} + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-z^2/2}\,dz$$

Thus the price for the ~~put~~ call option is

$$\vartheta(S_0,T) = S_0\,\Phi(d_+(T,S_0)) - e^{-rT}K\,\Phi(d_-(T,S_0))$$

"Black – Scholes formula".

We also have to consider non-continuous claims.

Theorem (portmanteau) 118: Let (M,d) be a metric space and $\mu, \mu_N \in \text{meas}(M,d)$. T.a.e.:

$1°$ $\mu_N \xrightarrow{w} \mu$

$2°$ $\forall_{f : M \longrightarrow \mathbb{R}}$ s.t. f is μ-a.s. continuous, $\mathcal{B}(M,d)$-measurable and bounded:

$$\int f \, d\mu_N \longrightarrow \int f \, d\mu.$$

Proof: Exercise.

Prop. 119: Let (M,d) be a metric space $\mu, \mu_n \in \text{meas}(M,d)$ s.t. $\mu_n \xrightarrow[n\to\infty]{w} \mu$

Let $f \in L^0(M, \mathcal{B}(M,d), \mu)$ be continuous μ-a.s.

Suppose $\exists_{p>1}$: $C = \sup_{n \in \mathbb{N}} \int |f|^p d\mu_n < \infty$

Then $\int f \, d\mu_n \longrightarrow \int f \, d\mu$.

(Here we don't require the boundedness anymore.)

<u>Proof</u>: W.l.o.g. $f \geq 0$ $k \in \mathbb{N}$

$f_k := f \wedge k$

portmanteau theorem

$\Rightarrow \int f_k \, d\mu_N \xrightarrow{N \to \infty} \int f_k \, d\mu.$

Further

$$\int (f-k)^+ d\mu_N \leq \int_{\{f > k\}} f \, d\mu_N$$

$$\leq \frac{1}{k^{p-1}} \int f^{p-1} f \, d\mu_N \leq \frac{C}{k^{p-1}}$$

$$\Rightarrow \int f_k \, d\mu_N \leq \int f \, d\mu_N \leq \int f_k \, d\mu_N + \frac{C}{k^{p-1}}$$

Take $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ s.t. $\frac{C}{k_0^{p-1}} < \varepsilon$

$$N \to \infty \Rightarrow \int f_k \, d\mu \leq \underline{\lim}_N \int f \, d\mu_N$$

$$\leq \overline{\lim}_N \int f \, d\mu_N \leq \int f_k \, d\mu + \varepsilon.$$

$k \to \infty$ and monoton convergence $\Rightarrow$ □

In Prop. 113 occurs an L^p-boundedness condition

We give for the Black-Scholes a condition on f which implies it.

End of Lecture 26. 26th of May 2022

<u>Prop 120:</u> Let $f:]0, \infty[\to \mathbb{R}$

be $\mathcal{B}(]0,\infty[)$-measurable, λ-a.s. continuous

and suppose that

$$|f(x)| \leq C(1+x)^q \quad \forall x \in]0, \infty[$$

for some $C > 0$ and $q \in [0, 2[$.

Then $\lim_{N \to \infty} E_{P_N^*}[f(S_N^{(N)})] = E_{P^*}[f(S_T)]$

(P^* is a measure for which $W_T \sim \mathcal{N}(0, T)$)

<u>Proof:</u>

$\ln E_{P_N^*}[(S_N^{(N)})^2] = 2 \ln S_0 +$

$$\ln \left(\prod_{k=1}^{N} \left(\mathrm{var}(1 + R_k^{(N)}) + (1 + r_N)^2 \right) \right)$$

$$= 2 \ln S_0 + \sum_{k=1}^{N} \ln\left(\mathrm{var}_N(1 + R_k^{(N)}) + (1 + r_N)^2 \right)$$

$$\leq 2|\ln S_0| + \sigma_N^2 T + |2 N r_N| + N r_N^2$$

$$+ \tilde{C} \Big(\underbrace{\sum_{k=1}^{N} \left(\mathrm{var}_N(1 + R_k^{(N)}) + 2|r_N| + |r_N|^2 \right)^2}_{\xrightarrow{N \to \infty} 0} \Big)$$

for some $\tilde{C} > 0$

$\Rightarrow \sup_N E_{P_N^*}[(S_N^{(n)})^2] < \infty$

Suppose $q \in]0,2[$ (The case $q=0$ is the bounded case.) Put $p = \frac{2}{q} > 1$

Then $\sup_N E_{P_N^*}[|f(S_N^{(n)})|^p]$

$$\leq C^p \sup_N E_{P_N^*}[(1+S_N^{(n)})^2] < \infty$$

Prop 119 $\Rightarrow$ $\square$

What about the price at $0 \leq t < T$?

$$H_N(\omega) = \frac{f(S_N^{(N)}(\omega))}{(1+r_N)^N}$$

<u>Remark 121:</u> $S_{\lfloor t \frac{N}{T} \rfloor} \xrightarrow{w} S_t$

$$S_t(\omega) = S_0 \exp\left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right)t\right)$$

(W_t) a Brownian motion.

Put $\mathcal{F}_k^{(N)} := \sigma(S_j^{(N)}, 0 \leq j \leq k)$

For the N-th market: The price at t is

$$V_t^{(N)} = E_{P_N^*}\left[\frac{f(S_N^{(N)})}{(1+r_N)^N} \,\middle|\, \mathcal{F}^{(N)}_{\lceil t \frac{N}{T} \rceil}\right]$$

This converges to

$E_{P^*}\left[\frac{f(S_T)}{e^{rT}} \,\middle|\, \tilde{\mathcal{F}}_t\right]$ in the following sense.

$$\mathcal{F}_t = \sigma(W_s \mid s \leq t) = \sigma(S_s \mid s \leq t)$$

and for all $A \in \mathcal{F}_t$ of the

form $\cdot\, A = S^{-1}(B)$,

$B = (C \times \mathcal{C}[t,T]) \cap \mathcal{C}[0,T]$ with $C \in \mathcal{B}(\mathcal{C}[0,t])$

$$\cdot\ P^*(S^{-1}(\partial B)) = 0$$

we have

$$E_{P_N^*}\left[E_{P_N^*}\left[\frac{f(S_N^{(N)})}{(1+r_N)^N} \,\Big|\, \mathcal{F}_{\lceil t \frac{N}{T} \rceil}\right] \mathbb{1}_{S^{(N)-1}(B)}\right]$$

$$\xrightarrow{N \to \infty} E_{P^*}\left[E_{P^*}\left[\frac{f(S_T)}{e^{rT}} \,\Big|\, \mathcal{F}_t\right] \mathbb{1}_A\right]$$

let us compute $E_{P^*}\left[\frac{f(S_T)}{e^{rT}} \,\Big|\, \mathcal{F}_t\right]$

$$\frac{S_T}{S_t} = \exp\left(\sigma(W_T - W_t) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$\overset{(*)}{\sim} \exp\left(\sigma W_{T-t} + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)$$

$$\Rightarrow E_{P^*}\left[\frac{f(S_T)}{e^{rT}} \,\Big|\, \mathcal{F}_t\right](\omega)$$

$$\underset{\uparrow}{=} E_{P^*}\left[\frac{f\left(S_t(\omega)\frac{S_T}{S_t}\right)}{e^{rT}}\right]$$

$\frac{S_T}{S_t}$ is independent to $\mathcal{F}_t$ (a fact)

$$\overset{(*)}{=} E_{P^*}\left[\frac{f\left(S_t(\omega)\cdot\frac{S_{T-t}}{S_0}\right)}{e^{(T-t)r}}\right] e^{-tr}$$

$$=: V_t(\omega)$$

We see that $V_t(\omega)$ has the form

$u(S_t(\omega), T-t)\cdot e^{-rt}$

where $u(x,s)$ = price of $f(S_s)$ at 0

with start $S_0 = x$.

More precisely:

Put $u(x,s) := E_{P^*}\left[\dfrac{f\left(x \frac{S_s}{S_0}\right)}{e^{rs}} \right]$

for $x > 0$ and $0 \le s \le T$.

Then $V_t(\omega) = u(S_t(\omega), T-t)\, e^{-rt}$.

<u>Prop. 122:</u> Let $f \in C(]0,\infty[, \mathbb{R})$ with $|f(x)| \le C(1+x)^p$ $\forall x > 0$ for one p s.t. $0 \le p$.

Then u satisfies the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = rx \dfrac{\partial u}{\partial x} + \dfrac{\sigma^2}{2} x^2 \dfrac{\partial^2 u}{\partial x^2} - ru \\ \text{with } \lim\limits_{t \downarrow 0} u(x,t) = f(x) \\ \text{(locally uniform) } \forall x \in]0,\infty[. \end{cases}$$

("Black-Scholes PDE")

We prove Prop 122:

Lemma 123: Consider f as in Prop 122.
and put $g(y) := f(e^y)$, $y \in \mathbb{R}$.
Then $j(x,t) := E_{P^*}[g(B_t + x)]$
($(B_t)_t$ a Brownian motion)
satisfies:

$$\frac{\partial j}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} j$$

and $\lim_{t \searrow 0} j(x,t) = g(x)$

Proof: Density for B_t: $\gamma_t(y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}$, $y \in \mathbb{R}$.

$$\Rightarrow \quad \frac{\partial \gamma_t(y)}{\partial t} = -\frac{1}{2t} \gamma_t(y) + \frac{y^2}{2t^2} \gamma_t(y)$$

and $\frac{\partial \gamma_t(y)}{\partial y} = -\frac{y}{t} \gamma_t(y)$; $\frac{\partial^2 \gamma_t(y)}{\partial y^2} = -\frac{\gamma_t(y)}{t} + \frac{y^2}{t^2} \gamma_t(y)$

So

$$\frac{\partial \gamma_t(y)}{\partial t} = \frac{1}{2} \frac{\partial^2 \gamma_t(y)}{\partial y^2}$$

$$j(x,t) = E_{P^*}[g(x+B_t)] = \int_{\mathbb{R}} g(x+y)\, \gamma_t(y)\, dy$$

$$= \int_{\mathbb{R}} g(y)\, \gamma_t(y-x)\, dy$$

$$\Rightarrow \frac{\partial j}{\partial t} = \frac{1}{2}\frac{\partial^2 j}{\partial x^2}$$

$$\lim_{t\to 0} j(x,t) = \lim_{t\downarrow 0} E_{P^*}[g(x+B_1\cdot\sqrt{t})]$$

$$\underset{\substack{\uparrow \\ \text{Lebesgue}}}{=} E_{P^*}[g(x)] = g(x). \qquad \square$$

<u>Proof</u> of Prop 122: Put $a := r - \frac{\sigma^2}{2}$.

$$u(x,t) = e^{-rt}\int_{\mathbb{R}} g(\sigma y + \ln x + at)\, \gamma_t(y)\, dy$$

$$= e^{-rt}\int_{\mathbb{R}} g(\sigma y + \ln x)\, \gamma_t(y - \tfrac{a}{\sigma}t)\, dy$$

$$\frac{\partial u}{\partial t} = -ru + e^{-rt}\frac{\partial^2 \int_{\mathbb{R}} \cdots dy}{\partial(\ln x)^2}\cdot\frac{\sigma^2}{2}$$

$$+ \frac{a}{\sigma}e^{-rt}\frac{\partial \int_{\mathbb{R}} \cdots dy}{\partial(\ln x)}\cdot\sigma$$

$$\frac{\partial^2 \psi}{\partial(\ln x)^2} = \frac{\partial}{\partial \ln x}\left(\frac{\partial \psi}{\partial(\ln x)}\right) = \frac{\partial}{\partial \ln x}\left(\frac{\partial \psi}{\partial x} \cdot x\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x} \cdot x\right) x$$

$$= x^2 \frac{\partial^2 \psi}{\partial x^2} + x \frac{\partial \psi}{\partial x}.$$

$$\frac{\partial \psi}{\partial(\ln x)} = \frac{\partial \psi}{\partial x} \cdot x$$

So $\frac{\partial u}{\partial t} = -ru + x^2 \sigma^2 \frac{\partial^2 u}{\partial x^2} + rx \frac{\partial u}{\partial x}.$

$\square$

Example 124: $f = (x-K)^+,\ K > 0$

Then $|f(x)| \leq |x| \quad (p=1)$

$$v(x,t) = x\,\Phi(d_+(x,t)) - Ke^{-rt}\,\Phi(d_-(x,t))$$

$$d_+(x,t) = d_-(x,t) + \sigma\sqrt{t}$$

$$d_-(x,t) = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

$$\frac{\partial v}{\partial t} = x\, e^{-\frac{d_+^2}{2}} \left(\frac{\partial d_-}{\partial t} + \frac{1}{2\sqrt{t}}\sigma \right)$$
$$- K e^{-rt} e^{-\frac{d_-^2}{2}} \cdot \frac{\partial d_-}{\partial t}$$
$$+ r K e^{-rt} \Phi(d_-(x,t))$$

(exercise $x\, e^{-\frac{d_+^2}{2}} - K e^{-rt} e^{-\frac{d_-^2}{2}} \overset{(**)}{=} 0$)

$$= x\, e^{-\frac{d_+^2}{2}} \frac{\sigma}{2\sqrt{t}} + rK e^{-rt} \Phi(d_-)$$

$$\frac{\partial v}{\partial x} \overset{(**)}{=} \Phi(d_+(x,t))$$

$$\frac{\partial^2 v}{\partial x^2} = e^{-\frac{d_+^2}{2}} \cdot \frac{1}{\sigma\sqrt{t}\, x}$$

$$\Rightarrow \frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} - rv + rx\frac{\partial v}{\partial x}$$

End of Lecture 27. 31.05.2022

III American contingent claims

We are given a multi-period market model $(\underline{S}_*, \tilde{\mathcal{F}}_* \subseteq \underset{\mathcal{F}}{\overset{\shortparallel}{\mathcal{F}_T}}, P)$ with P trivial on $\mathcal{F}_0$.

Recall A1: A non-negative adapted process $(C_t)_{t=\overline{0,T}}$ is called American contingent claim.

The process $H_t := \frac{C_t}{S_t^0} \overset{t=\overline{0,T}}{\vee}$ is call a "discounted American clam".

Examples A2: a) European cont. claims are Am. cont. claims in the following way:

C_T a Europ. cont. claim.

Put $C_0 = \ldots = C_{T-1} \equiv 0$.

b) American call option with strike K:

$(C_t)_{t=\overline{0,T}}$ with $C_t := (S_t^{(1)} - K)^+$

— Am 02 —

c) Bermuda options.

(In "between" Europ. and Am. cont. claims)

$\tilde{\mathcal{T}} \subseteq \{0, \dots, T\}$

Can be exercised at the times in $\tilde{\mathcal{T}}$.

For example a Bermuda call option

$$(C_t)_{t=\overline{0,T}} \qquad C_t := \begin{cases} 0, & t \notin \tilde{\mathcal{T}} \\ (S_t^{(1)} - K)^+, & t \in \tilde{\mathcal{T}}. \end{cases}$$

<u>Questions A3:</u>

- Seller perspective: How to superhedge an American contingent claim?
- Buyer's perspective: How to optimize the exercise strategy?
- market perspective:
  What is an AF price for C and can one compute such a price?

III.1. Seller's perspective

The seller of an Am. cont. claim wants to hedge against the claim, i.e. needs an adapted process $(U_t)_{t=\overline{0,T}}$ such that U_t is given by a self-financing trading strategy and

$$U_t \geq H_t \quad \forall t = 0, \dots, T$$

Assumption A4: $\mathcal{P} = \{P^*\}$, i.e. the MM is AF and complete.

We need to find the "price" which we need to invest to hedge a discounted Am. claim $H = (H_t)_{t=\overline{0,T}}$.

We do this inductively backwards.

$t = T$: $\quad U_T = H_T$

$t < T$: Suppose $U_{t+1}, \dots, U_T$ are defined such that

— Am 04 — $\forall s = t+1, \dots, T-1$:

. $U_s \geq H_s$ and

. $U_s \geq E^*[U_{s+1} | \mathcal{F}_s]$

(We need to cover the price for U_{s+1} !)

Put $U_t = H_t \vee E^*[U_{t+1} | \mathcal{F}_t]$.

Def A5: The process $U = (U_t)_{t=\overline{0,T}}$ is called the "Snell envelope of H".

Prop A6: $(U_t)_{t=\overline{0,T}}$ is a supermartingale and every supermartingale $(\tilde{U}_t)_{t=\overline{0,T}}$ such that $\tilde{U}_t \geq H_t \ \forall t=\overline{0,T}$ must satisfy $\tilde{U}_t \geq U_t \ \forall t=\overline{0,T}$.

Proof: $U_t \geq E^*[U_{t+1} | \mathcal{F}_t]$ by construction.

(Supermartingale: we have more than we need to hedge the next payoff (in a complete market))

Induction on t: $\tilde{U}_T \geq H_T = U_T$

and for $t < T$ we have

$$\tilde{U}_t \underset{\substack{\uparrow \\ \text{supermartingale}}}{\geq} E^*[\tilde{U}_{t+1} | \mathcal{F}_t] \geq E^*[U_{t+1} | \mathcal{F}_t]$$

and $\tilde{U}_t \geq H_t$.

$$\Rightarrow \quad \tilde{U}_t \geq H_t \vee E^*[U_{t+1} | \mathcal{F}_t] = U_t. \quad \square$$

Further U gives you at every time the best price for the hedging of H.

<u>Proposition A7:</u> 1) $\exists$ a predictable process $(\xi_t)_{t=\overline{1,T}}$ such that

$$\forall_{t=\overline{0,T}} \; \forall_{u \geq t} : \quad U_t + \sum_{s=t+1}^{u} \xi_s (X_s - X_{s-1}) \geq H_u.$$

2) Let $\tilde{U}_t \in L^0(\Omega, \mathcal{F}_t, P)$ such that $\exists$ a predictable process $(\xi_u)_{u=\overline{t+1,T}}$ such that

— Am 06 —

$\forall_{u=t,\dots,T}: \tilde{U}_t + \sum_{s=t+1}^{u} \xi_s (X_s - X_{s-1}) \geq H_u$

Then $\tilde{U}_t \geq U_t$.

(i.e. U gives at every time the price for an optimal hedge)

Proof: The existence of ξ follows from the Doob decomposition theorem and the martingale representation property (Prop. 10.7, 4°)

We prove the second assertion:

Note that $\tilde{U}_t$ and the ξ_s are bounded, because $\mathcal{F} = \mathcal{F}_T$ is generated by finitely many atoms (up to zero sets), by completeness.

Induction on t: $\underline{t=T:}$ $\tilde{U}_T \geq H_T = U_T$ ✓

$\underline{t<T}$: $\tilde U_t + \xi_{t+1}(X_{t+1} - X_t) \geq U_{t+1}$

by (IH).

$$\Rightarrow \tilde U_t = E^*[\tilde U_t + \xi_{t+1}(X_{t+1} - X_t) | \mathcal{F}_t]$$
$$\geq E^*[U_{t+1} | \mathcal{F}_t]$$

$$\underset{\tilde U_t \geq H_t}{\Rightarrow} \quad \tilde U_t \geq U_t \qquad \square$$

III.2. The ~~[illegible]~~ Gaspar's perspective — Am 0-8 —

We want to find an optimal exercise strategy.

<u>Our setting:</u> ($\underline{S}_t$, $\mathcal{F}_t$ with $\mathcal{F}_T = \mathcal{F}$, P) multi-period s.t. P is trivial on $\mathcal{F}_0$

- We assume that the market is AF and complete.
- $(C_t)_{t \in \{0,\dots,T\}}$ a American (contingent) claim, $H_t = C_t \cdot \frac{1}{S_t^{(0)}}$, $t = 0,\dots,T$, the discounted American claim.

Recall that we have 3 perspectives, see A3.

<u>Def A 8:</u> $\tau : \Omega \longrightarrow \{0,1,\dots,T\} \cup \{+\infty\}$ is called stopping time if $\forall_{t=\overline{0,T}} : \{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{F}_t$

We can observe when the ~~[illegible]~~ time ~~[illegible]~~ stops.

<u>Remark A 9:</u> If τ and σ are stopping times then $\tau \wedge \sigma$ and $\tau \vee \sigma$ and $\tau + \sigma$ are stopping times. (exercise)

Example A.10: Let $(Y_t)_{t=\overline{0,T}}$ be an adapted process and $c \in \mathbb{R}$.

$$\tau(\omega) := \inf\{t \geq 0 \mid Y_t(\omega) \geq c\}$$

is a stopping time.

Proof: $\{\tau \leq s\} = \{Y_0 \geq c\} \cup \{Y_1 \geq c\} \cup \dots$
$\cup \{Y_s \geq c\} \in \mathcal{F}_s$. □

Def A.11: Let $(Y_t)_{t=\overline{0,T}}$ be a process. Put

$Y_t^\tau(\omega) := Y_{t \wedge \tau(\omega)}(\omega)$. Y_*^τ is called the process stopped in τ.

Thm A12 (Doob's stopping theorem)

Let M be an adapted process, $M_t \in L^1(Q)$

T.a.e.:

1° M is a Q-martingale

2° $\forall$ stopping times τ: M^τ is a Q-martingale

3° $\forall$ — " — : $E_Q[M_{\tau \wedge T}] = E_Q[M_0]$

Proof: At first note that $M_t^\tau \in L^1(\Omega, \mathcal{F}_t, Q)$:

- $\mathcal{F}_t$-measurable: $B \in \mathcal{B}(\mathbb{R})$.

$$\{M_t^\tau \in B\} = \left(\{\tau > t\} \cap \{M_t \in B\}\right) \cup \bigcup_{s=0}^{t} \left(\{\tau = s\} \cap \{M_s \in B\}\right)$$

$$\in \mathcal{F}_t.$$

- Q-integrable: $|M_t^\tau| \leq \sum_{s=0}^{t} |M_s| \in L^1(Q)$

$1° \Rightarrow 2°$: $M_{t+1}^\tau = M_{t+1} \cdot \mathbb{1}_{\{\tau > t\}} + M_t^\tau \cdot \mathbb{1}_{\{\tau \leq t\}}$

$$\Rightarrow E_Q[M_{t+1}^\tau \mid \mathcal{F}_t] = E_Q[M_{t+1} \mid \mathcal{F}_t] \mathbb{1}_{\{\tau > t\}} + M_t^\tau \mathbb{1}_{\{\tau \leq t\}}$$

$$= M_t \mathbb{1}_{\{\tau > t\}} + M_{t \wedge \tau} \cdot \mathbb{1}_{\{\tau \leq t\}}$$

$$= M_{t \wedge \tau} = M_t^\tau.$$

$2° \Rightarrow 3°$: $M_0 = E_Q[M_T^\tau \mid \mathcal{F}_0]$. So they have the same expectation.

$3° \Rightarrow 1°$: Take $A \in \mathcal{F}_t$. $\tau(\omega) := \begin{cases} t, & \omega \in A \\ T, & \omega \notin A. \end{cases}$

— Anal — 11 —

Then τ is a stopping time:

$s \neq t, T$: $\{\tau = s\} = \phi$

$\{\tau = t\} = A \in \mathcal{F}_t$

$\{\tau = T\} = \Omega \setminus A. \in \mathcal{F}_t \subseteq \mathcal{F}_T.$

Then $E_Q[M_0] = E_Q[M_t \mathbb{1}_A] + E_Q[M_T \mathbb{1}_{\Omega \setminus A}]$

$= -E_Q[(M_T - M_t)\mathbb{1}_A] + E_Q[M_T]$

$\sigma \equiv T$ is also a stopping time, so $E_Q[M_T]$

$= E_Q[M_0]$

Therefore: $0 = E_Q[(M_T - M_t)\mathbb{1}_A]$ $\square$

<u>Corollary (6.18) A13</u>: Let U be an adapted process such that $U_t \in L^1(Q)$ for all $t = \overline{0, T}$

T.a.e.: 1° U is a Q-supermartingale

2° $\forall$ stopping times τ: U^τ is a Q-supermartingale.

<u>Proof</u>: <u>$2° \Rightarrow 1°$</u>: Take $\tau \equiv T$.

<u>$1° \Rightarrow 2°$</u>: $U = M - A$ A predictable, $A_0 \equiv 0$ and A increasing.

$U^\tau = M^\tau - A^\tau$. So by Doob's stopping theorem U^τ is a Q-supermartingale. □

We want to have an optimal exercise time for the discounted American claim, i.e.

$$\tau \in \mathcal{T} := \{ \sigma \mid \sigma \text{ stopping time with } \sigma \leq T \}$$

such that $E_{P^*}[H_\tau]$ is maximal among all $\tau \in \mathcal{T}$.

We can ask this question for any measure equivalent to P

Remark A14: P can be preference measure for the ~~buy~~ buyer coming from a utility function U, say P is chosen such that $E_P[U(X_t)] = U(X_0)$ and

$$E_P[U(X_t) \mid \mathcal{F}_s] = U(X_s) \ \forall_{s<t}$$

if possible.

Then one maximizes $E[U(H_\tau)]$ on $\mathcal{T}$. Write $\tilde{H}_t = U(H_t)$.

Then we have to maximize $E[\tilde{H}_t]$.

$\rightsquigarrow$ We get the question using P instead of P^*

Let us skip A.F. first and assume $H_t \in \mathcal{L}^1(P)$, $t = \overline{0,T}$.

<u>Def A15:</u> A stopping time τ^* is called ~~opt~~ optimal (for H w.r.t. P)

if $$E[H_{\tau^*}] = \sup_{\tau \in \mathcal{T}} E[H_\tau]$$

We construct an optimal stopping time.

Let U^P be the Snell envelope of H w.r.t. P. We define:

$$\tau_{min} = \min\{t \geq 0 \mid U_t = H_t\}$$

It is an element of $\mathcal{T}$ because $U_T = H_T$.

end of Lecture 28, 2.6.22

<u>Prop. A.16:</u>
τ_{min} is an optimal stopping time.

Prop. A16 is a consequence of the more general: Theorem A17. For that we need $\mathcal{T}_t := \{\tau \in \mathcal{T} \mid \tau \geq t\}$ and $\tau_{min}^{(t)} := \min\{s \geq t \mid U_s = H_s\}$.

Theorem A.17: (Thm 6.20.)

We have

$$U_t = E\left[H_{\tau_{min}^{(t)}} \mid \mathcal{F}_t \right] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} E[H_\tau \mid \mathcal{F}_t]$$

in particular

$$U_0 = \sup_{\tau \in \mathcal{T}} E[H_\tau] \quad (P \text{ is trivial on } \mathcal{F}_0)$$

Excurs: Essential supremum.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let Φ be a set of random variables with values in $[-\infty, \infty]$.

Def A.18: A random variable φ^* with values in $[-\infty,\infty]$ with the properties

φ^* (i) $\varphi^* \geq \varphi$ P-a.s. $\forall \varphi \in \Phi$

(ii) For every other random variable ψ with values in $[-\infty,\infty]$ satisfying ψ(i) we have $\psi \geq \varphi^*$ P-a.s.

is called an "essential supremum of Φ, denoted by $\operatorname{esssup} \Phi$ or $\operatorname{esssup}_{\varphi \in \Phi} \varphi$.

Prop A19 (Theorem A.18 in the textbook)

(a) $\operatorname{esssup} \Phi$ exists.

(b) Suppose Φ is directed, i.e. $\forall_{\varphi_1, \varphi_2 \in \Phi}$: $\exists\, \varphi \in \Phi$: $\varphi \geq \varphi_1 \vee \varphi_2$ P-a.s.

Then $\exists\ \varphi_1 \underset{P\text{-a.s.}}{\leq} \varphi_2 \underset{P\text{-a.s.}}{\leq} \varphi_3 \underset{P\text{-a.s.}}{\leq} \dots$ in Φ:

$\lim_{n \to \infty} \varphi_i = \operatorname{esssup} \Phi$ P-a.s.

Proof: W.l.o.g. we can assume that $\forall_{\varphi \in \Phi}$ $\operatorname{im} \varphi \subseteq [-1, 1]$.

(If not then consider

$$f: [-\infty, \infty] \xrightarrow{\sim} [-1, 1]$$

$$f(x) = \begin{cases} 1, & x = \infty \\ -1, & x = -\infty \\ \operatorname{sgn}(x) \cdot (1 - e^{-x^2}), & x \notin \{\infty, -\infty\}. \end{cases}$$

— AmO-16 —
and study $f \circ \Phi = \{ f \circ \varphi \mid \varphi \in \Phi \}$.)

For $\Psi \subseteq \Phi$ countable the map $\varphi_\Psi :=$

$\sup_{\varphi \in \Psi} \varphi$ is measurable.

Take $\Psi^* \subseteq \Phi$ countable such that

$$E[\varphi_{\Psi^*}] = \sup \{ E[\varphi_\Psi] \mid \Psi \subseteq \Phi \text{ countable} \}$$

Put $\varphi^* := \varphi_{\Psi^*}$

φ^* satisfies (i): Take $\varphi \in \Phi$

$\underbrace{\Psi^* \cup \{\varphi\}}_{=: \Psi}$ is countable.

Then $\varphi_\Psi \geq \varphi^*$ P-a.s. and

$E[\varphi_\Psi] = E[\varphi^*]$. So $\varphi_\Psi = \varphi^*$

P-a.s. $\Rightarrow$ $\varphi^* = \varphi_\Psi \geq \varphi$ P-a.s.

φ^* satisfies (ii): Let $\tilde{\varphi}$ be a random var. with values in $[-1,1]$ s.t.

$\tilde{\varphi} \geq \varphi$ P-a.s. $\forall \varphi \in \Phi$.

$\Rightarrow \tilde{\varphi} \geq \varphi_{\Psi^*} = \varphi^*$ P-a.s.

For assertion (2) take for

$\Psi^* = \{\varphi_1, \varphi_2, \varphi_3, \dots\}$ a sequence

$\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3, \dots,$ such that $\tilde{\varphi}_1 = \varphi_1$ and

$\tilde{\varphi}_i \geq \tilde{\varphi}_{i-1} \vee \varphi_i$ for all $i = 2, 3, \dots$.

Then $\lim_{i \to \infty} \tilde{\varphi}_i = \sup \tilde{\varphi}_i = \varphi^*$. □

Example A20: $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$

$$P = \lambda|_{\mathcal{F}}.$$

$\Phi = \{ \mathbb{1}_{\{x\}} \mid x \in [0,1] \}$.

Then $\operatorname{esssup} \Phi = 0$ λ-a.s.

and $\sup_{\varphi \in \Phi} \varphi = \mathbb{1}_{[0,1]}$.

Proof of Theorem A.17:

U is a supermartingale. Cor. A13 $\Rightarrow$

$U^{\tau^{(1)}_{min}}$ is a supermartingale $\Rightarrow$

$\cancel{U_1 \geq E[U^{\tau^{(1)}_{min}}_T \mid \mathcal{F}_1] = E[U_{\tau^{(1)}_{min}} \mid \mathcal{F}_1]}$

— 10 – 18 —

$\forall \tau \in \mathcal{T}_t$:

$$U_t \geq E[U^\tau_T \mid \mathcal{F}_t] \underset{\substack{\uparrow \\ \tau \leq T}}{=} E[U_\tau \mid \mathcal{F}_t]$$

$$\geq E[H_\tau \mid \mathcal{F}_t]. \quad P\text{-as.}$$

So $U_t \geq \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} E[H_\tau \mid \mathcal{F}_t]$ P-as.

We want equality and for that we show

$$U_t = E[H_{\tau^{(t)}_{min}} \mid \mathcal{F}_t] \underset{\substack{\uparrow \\ H_{\tau^{(t)}_{min}} = U_{\tau^{(t)}_{min}}}}{=} E[U_{\tau^{(t)}_{min}} \mid \mathcal{F}_t]$$

i.e. that $\left(U^{\tau^{(t)}_{min}}_s\right)_{s=\overline{t,T}}$ is a P-martingale w.r.t. $(\mathcal{F}_s)_{s=\overline{t,T}}$

Take $s \in \{t, t+1, \dots, T\}$.

On $\{\tau^{(t)}_{min} \geq s+1\}$ we have

$$U^{\tau^{(t)}_{min}}_s = U_s = E[U_{s+1} \mid \mathcal{F}_s] \vee H_s$$

$$\underset{\substack{\uparrow \\ U_s \neq H_s}}{=} E[U_{s+1} \mid \mathcal{F}_s] = E[U^{\tau^{(t)}_{min}}_{s+1} \mid \mathcal{F}_s]$$

On $\{\tau_{min}^{(A)} \leq s\}$:

$$U_s^{\tau_{min}^{(A)}} = U_{\tau_{min}^{(A)}} = U_{s+1}^{\tau_{min}^{(A)}}$$

In particular $U_{s+1}^{\tau_{min}^{(A)}} \cdot \mathbb{1}_{\{\tau_{min}^{(A)} \leq s\}}$

is $\mathcal{F}_s$ - measurable.

So we have

$$U_s^{\tau_{min}^{(A)}} = U_s^{\tau_{min}^{(A)}} \mathbb{1}_{\{\tau_{min}^{(A)} \geq s+1\}} + U_s^{\tau_{min}^{(A)}} \mathbb{1}_{\{-\shortparallel- \leq s\}}$$

$$= E\left[U_{s+1}^{\tau_{min}^{(A)}} \mathbb{1}_{\{-\shortparallel- \geq s+1\}} \middle| \mathcal{F}_s\right] + E\left[U_{s+1}^{\tau_{min}^{(A)}} \mathbb{1}_{\{-\shortparallel- \leq s\}} \middle| \mathcal{F}_s\right]$$

$$= E\left[U_{s+1}^{\tau_{min}^{(A)}} \middle| \mathcal{F}_s\right] \qquad \square$$

— 4.10.20 —
There are more optimal stopping times

$$\tau_{max} := \inf(\{t \geq 0 \mid E[U_{t+1} - U_t \mid \mathcal{F}_t] \neq 0\} \cup \{T\})$$

$$= \inf(\{t \geq 0 \mid A_{t+1} \neq 0\} \cup \{T\})$$

$\uparrow$

$U_t = M_t - A_t$ Doob decomposition.

<u>Theorem A21:</u> A stopping time τ is optimal iff $\tau \leq \tau_{max}$ and $U_\tau = H_\tau$. In particular τ_{max} is optimal.

<u>Proof:</u> (*) At first note: $\tau \in \mathcal{T}$ is optimal iff $H_\tau = U_\tau$ and U^τ is a P-martingale.

<u>Proof:</u> "$\Rightarrow$" τ optimal $\Rightarrow$

$$U_0 = E[H_\tau] \leq E[U_\tau] = E[U^\tau_T] \leq U_0$$

So U^τ is a supermartingale with $E[U^\tau_T] = U_0$

$\Rightarrow$ U^τ is a martingal.

Further $E[H_\tau] = E[U_\tau] \in \mathbb{R}$ and $H_\tau \leq U_\tau$ imply $H_\tau = U_\tau$ P-a.s.

"$\Leftarrow$" $U_0 \underset{\substack{\uparrow\\ \text{martingale}}}{=} E[U_T^\tau] = E[U_\tau] \underset{\substack{\uparrow\\ U_\tau = H_\tau}}{=} E[H_\tau]$ □

Now we start to prove the assertion of the theorem.

"$\Rightarrow$" $(*) \Rightarrow U_\tau = H_\tau$ and $U^\tau = M^\tau - A^\tau$ is a martingale, i.e. A^τ is a martingale, i.e. $A_T^\tau = 0$ P-as., i.e. $\tau \le \tau_{max}$ P-as.

"$\Leftarrow$" $A^\tau = 0$ P-as., because $\tau \le \tau_{max}$.
$\Rightarrow U^\tau = M^\tau$ P-as., i.e. U^τ is a martingale. $\underset{(*)}{\Rightarrow}$ τ is optimal.

Still need to prove that τ_{max} satisfies $U_{\tau_{max}} = H_{\tau_{max}}$.

On $\{\tau_{max} = T\}$: $U_T = H_T$ ✓

On $\{\tau_{max} = t\}$ for $t < T$:

$$E[U_{t+1} - U_t \mid \mathcal{F}_t] = -(A_{t+1} - A_t) = -A_{t+1} < 0$$

$$\Rightarrow U_t = H_t \text{ on } \{\tau_{max} = t\}$$ □

— AmO — 22 —

Now we get back to the complete market with $\mathbb{P} = \{P^*\}$.

We consider the P^*-Snell envelope of H: U^*. Then

U_0^* is the minimal amount to superhedge H and

the maximal expected amount one can get with an exercise strategy.

$\Rightarrow$ U_0^* is the unique AF price of H.

<u>Remark A 22:</u> (1) $\tau \equiv T$ is an optimal stopping time (w.r.t. P^*) iff U is a P^*-martingale iff $\forall_t: H_t \leq E^*[H_T \mid \mathcal{F}_t]$

In this case we have

$$\begin{array}{ccc} \pi(H) & = & \pi(H_T). \\ \shortparallel & & \shortparallel \\ \pi(C) & & \pi(C_T). \end{array}$$

(2) Suppose $H_t = f(X_t)$, $f: \mathbb{R}_+^d \longrightarrow [0,\infty)$.

convex.

$$\Longrightarrow \quad E^*[H_T \mid \mathcal{F}_t] \underset{\substack{\uparrow \\ \text{Jensen}}}{\geq} f(E^*[H_T \mid \mathcal{F}_t])$$

$$= f(X_t) = H_t.$$

$\Longrightarrow \quad \pi(H) = \pi(H_T)$

(3) Given the CRR-model for $a<r<b$.

$f = (x-K)^+ \quad , K>0.$

Suppose $r \geq 0$.

Then

$$E^*\left[\left(X_{t+1} - \frac{K}{(r+1)^{t+1}}\right)^+ \Big| \mathcal{F}_t\right]$$

$$\geq \left(X_t - \frac{K}{(r+1)^{t+1}}\right)^+$$

$$\geq \left(X_t - \frac{K}{(r+1)^t}\right)^+$$

So $(H_t)_{t=\overline{0,T}}$ is a P^* submartingale

$\Longrightarrow \quad \tau \equiv T$ is optimal and

H and H_T (European discounted claim) have the same price.

AmO — 24

(4) Suppose we are given the CRR model with $-1 < a < 0 < r < b$ and we consider the American put.

$$C := (K - S_t)_{t=\overline{0,T}}.$$

<u>Case 1:</u> $S_0 \geq \frac{K}{(1+a)^T} \Rightarrow \pi^C(S_0) = 0$

$$= (K - S_0)^+.$$

<u>Case 2:</u> $S_0 \leq \frac{K}{(1+b)^T}$

$$\pi^C(S_0) = \sup_{\tau \in \mathcal{T}} \left(E^*\left[\frac{K}{(1+r)^\tau}\right] - S_0 \right)$$

$$\underset{\tau \equiv 0}{=} \quad K - S_0 = (K - S_0)^+$$

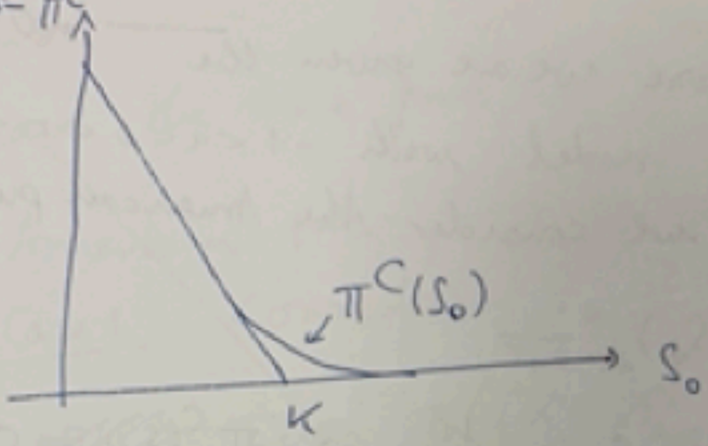
(In both cases we get the intrinsic value at $t=0$, i.e. $(K-S_0)^+$.)

<u>Case 3:</u> $K \leq S_0 < \frac{K}{(1+a)^T}$

$$\Rightarrow \pi^C(S_0) > 0 = (K - S_0)^+$$

So $\pi^C(S_0)$ is not the intrinsic value at $t=0$.

—AmD-25—



($\pi^c(S_0) \geq$ intrinsic value, because $\tau \equiv 0$ is a stopping time $\in \mathcal{T}$.)

Little exercise:

$$\exists\ x^* \in \left[\frac{K}{(1+b)^T}, \frac{K}{(1+a)^T}\right):$$

$$\pi^c(x) = (K-x)^+ \ , 0<x \leq x^*$$

$$\pi^c(x) > (K-x)^+ \ , \text{ for } x^* < x < \frac{K}{(1+a)^T}$$

$$\pi^c(x) = 0 \qquad , \text{ for } x \geq \frac{K}{(1+a)^T}.$$

<u>Homework:</u> Study Remark 6.29 in the textbook.

End of Lecture 29. 07.06.22

III §3 Arbitrage free prices for American claims

We drop the completeness condition.

$\tau \in \mathcal{T}$ $\quad \Pi(H_\tau) = \left\{ E_Q[H_\tau] \;\middle|\; Q \in \mathcal{P} \text{ and } E_Q[H_\tau] < \infty \right\}$

<u>Def 23</u>: Let $\pi \in \mathbb{R}$ is called an Arbitrage free price for H if

(i) $\exists\, \tau \in \mathcal{T} : \pi \in \Pi(H_\tau)$
(the buyer's demand)

(ii) $\forall_\tau \in \mathcal{T}\ \exists\, \tilde{\pi} \in \Pi(H_\tau) : \tilde{\pi} \leq \pi$.
(the seller's demand)

$\Pi(H) = \{ \pi \in \mathbb{R} \mid \pi \text{ is an arbitrage-free price for } H \}$

$\pi^{\downarrow}(H) := \inf \Pi(H)$
$\pi^{\uparrow}(H) := \sup \Pi(H)$.

Assumption A24: $E^*[H_t] < \infty$

$$\forall_{t \in \{0,\dots,T\}} \forall_{P^* \in \mathcal{P}}$$

We have boundaries:

$$\sup_{\tau \in \mathcal{T}} \inf_{P^* \in \mathcal{P}} E^*[H_\tau] \le \pi \le \sup_{\tau \in \mathcal{T}} \sup_{P^* \in \mathcal{P}} E^*[H_\tau]$$

for all $\pi \in \Pi(H)$ by definition.

Theorem A25: Under A24

(i) $\Pi(H) \neq \emptyset$.

(ii) $\pi^{\downarrow}(H) = \inf_{P^* \in \mathcal{P}} \sup_{\tau \in \mathcal{T}} E^*[H_\tau] \overset{(*)}{=} \sup_{\tau \in \mathcal{T}} \inf E^{Q}[H_\tau]$

(iii) $\pi^{\uparrow}(H) = \sup_{\tau \in \mathcal{T}} \sup_{P^* \in \mathcal{P}} E^*[H_\tau]$.

(iv) $\Pi(H)$ has exactly one element if the market is complete

(v) $\Pi(H)$ is an intervall which does not contain $\pi^{\uparrow}(H)$ if ~~the market is not complete~~ $|\Pi(H)| \neq 1$

<u>Proof</u>: The equality (*) is non-trivial.
$\geq$ ✓ , $\leq$ is more difficult and part of Theorem 6.41 in the textbook.

- (i) Let $\tau^* \in \mathcal{T}$ be an optimal stopping time w.r.t. a $P^* \in \mathcal{P}$.
Then $U_0^* = E^*[H_{\tau^*}] \in \Pi(H)$.
Thus $\{U_0^Q \mid Q \in \mathcal{P}\} \subseteq \Pi(H)$.

$\Rightarrow$ (ii) and (iii).

- $\Pi(H)$ is convex, because $\{U_0^Q \mid Q \in \mathcal{P}\}$ is convex: Take $P_1, P_2 \in \mathcal{P}$ and $\alpha \in [0,1]$.

$$f_\tau : [0,1] \longrightarrow \mathbb{R}$$

$$f_\tau(\alpha) := \alpha E_{P_1}[H_\tau] + (1-\alpha) E_{P_2}[H_\tau]$$

is an affine map and its graph is a segment. Put $P_\alpha := P_1 \alpha + (1-\alpha) P_2$

$$f : [0,1] \longrightarrow \mathbb{R} \quad f(\alpha) = U_0^{P_\alpha}$$

Then $f = \sup_{\mathcal{I}} f_\tau$.

$\Rightarrow$ f is convex and lower semicontinuous, thus continuous.

Intermediate value theorem

$\Rightarrow$ $\operatorname{im} f = [\inf f, \sup f]$

(iv) and (v):

Let us now assume that $\tilde{\pi} = \pi^\uparrow(H) \in \Pi(H)$.

Then $\exists \tau \in \mathcal{T}$ $\exists P^* \in \mathcal{P}$: $\tilde{\pi} = E^*[H_\tau]$

$$= \sup_{\sigma \in \mathcal{T}} \sup_{Q \in \mathcal{P}} E_Q[H_\sigma].$$

In particular $E_{P^*}[H_\tau] \in \Pi(H_\tau)$

H_τ can be interpreted as a discounted European claim with expiry date T.

Prop 99(b) $\Rightarrow$ H_τ is replicable and so $E_{P^*}[H_\tau] = E_Q[H_\tau]$ $\forall Q \in \mathcal{P}$.

$$\Rightarrow \pi^\uparrow(H) = \sup_{\sigma \in \mathcal{T}} \inf_{Q \in \mathcal{P}} E_Q[H_\sigma] = \pi^\downarrow(H)$$

$\Rightarrow$ $\Pi(H) = \{\tilde{\pi}\}$. □

Example A.26:

(i) Consider the market

$$(\underline{S}_{*}, \mathcal{F}_0 = \mathcal{F}_1 = \mathcal{F}_2 = \{\Omega, \emptyset\}, P), \Omega = \{\omega_0\}$$

and constant vector $\underline{S}_{*}$, i.e.

$$\binom{1}{1} = \underline{S}_0 = \underline{S}_1 = \underline{S}_2 \in \mathbb{R}^2$$

$$\tilde{\Omega} := \{(\omega_0, +1), (\omega_0, -1)\}, \quad \tilde{P}((\omega_0, \varepsilon)) = \frac{1}{2}.$$

$$\tilde{\mathcal{F}}_0 := \{\tilde{\Omega}, \emptyset\} = \tilde{\mathcal{F}}_1 \subsetneqq \tilde{\mathcal{F}}_2 = \mathcal{P}(\tilde{\Omega}).$$

Consider $H = (H_0, H_1, H_2)$

$$H_0 \equiv 0, \quad H_1 \equiv 1, \quad H_2(\omega) = \begin{cases} 2, & \text{if } \omega = (\omega_0, 1) \\ 0, & \text{if } \omega = (\omega_0, -1). \end{cases}$$

Then $\tilde{\mathcal{P}} = \{ P^*_\lambda \mid \lambda \in]0, 1[\}$

$$P^*_\lambda(\{(\omega_0, 1)\}) = \lambda.$$

and ~~the~~ an optimal stopping time for P^*_λ and H is $\quad T_\lambda \equiv 2$ if $\lambda \geq \frac{1}{2}$

and $I_1 \equiv 1$ if $A < \frac{1}{2}$

$\Rightarrow \Pi^{\downarrow}(H) = 1 \leq 2 = \Pi^{\uparrow}(H)$

and $\Pi(H) = [1, 2)$, because

$1 = E_{P^*_{\frac{1}{4}}}[H_1] \in \Pi(H_1)$

and for $I \equiv 2$ we have

$$E_{P^*_{\frac{1}{4}}}[H_2] = \frac{1}{2} < 1.$$

$$\left(E_{P^*_{\frac{1}{4}}}[H_1] = 1 \leq 1 \right.$$

$$\left. E_{P^*_{\frac{1}{4}}}[H_0] = 0 \leq 1 \right)$$

(ii) If we replace in (i) H_1 by $H_1 \equiv 0$.

then $\Pi(H) = (0, 2)$.

<u>Def A 27:</u> A discounted American claim is called attainable if $\exists$ self-financing trading strategy $(\xi_t)_{t=\overline{1,T}}$, such that its value

$(V_t)_{t=\overline{0,T}}$ satisfies $V_t \geq H_t$ for all $t \in \{0,1,2,\dots,T\}$ and $\exists \tau \in \mathcal{T}$:

$V_\tau = H_\tau$.

<u>Remark A28:</u> 1) If $\mathcal{P} = \{P^*\}$, i.e. the market is complete, then every American claim is attainable and its unique AF price is U_0^*.

2) If $\mathcal{P}$ has more than one element then every ~~H~~ attainable H satisfies $E^*[H_t] < \infty$ $\forall P^* \in \mathcal{P}$ and $\forall t \in \{0,\dots,T\}$, and H has a unique A.F. price.

<u>Theorem A29:</u> T.f.a.a.e.: (under Ass. A.25)

1° H is attainable

2° $\#\Pi(H) = 1$

3° $\pi^\uparrow(H) \in \Pi(H)$.

<u>Proof:</u> $1° \overset{A28}{\Longrightarrow} 2° \overset{A25}{\Longleftrightarrow} 3°$;

$3° \Rightarrow 1°$ needs more technics: Remark 7.10 □

End of Lecture 30, 09.06.2022.