Problem Sheet 1: (15.2.22)

Problem 1) (10 pts) Given example of $(\underline{\Pi}, \underline{S})$ with ~~[illegible]~~ two probability spaces $(\Omega, \mathcal{F}, P_1)$, $(\Omega, \mathcal{F}, P_2)$ s.t.

$(\underline{\Pi}, \underline{S}, P_1)$ is arbitrage free and

$(\underline{\Pi}, \underline{S}, P_2)$ is not arbitrage free.

Problem 2) (10 pts.) Let ν be a Borel measure on $[0, \infty[$. Show that

F_ν is continuous if $\nu \ll \lambda|_{\mathcal{B}(\mathbb{R})}$.

— Problem Sheet 2 (22.2.22)

① (10 points) Suppose $(\Omega, \mathcal{F})$ is a measurable space with two probability measures P_1, P_2.
Suppose $\mathcal{F}$ is finitely generated, i.e.
$\exists\, S \subseteq \mathcal{P}(\Omega) : \sigma(S) = \mathcal{F}$ and $|S| < \infty$.

(a) Find a sufficient and nessecary condition such that there exist a Radon–Nikodym density $\frac{dP_2}{dP_1}$

(b) Give an explicit formula for $\frac{dP_2}{dP_1}$

② (10 points) Suppose we are given a probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ $(|\Omega| = 3)$
and $P(\{\omega_j\}) > 0$, $j = 1, 2, 3$.
We consider a market model:

i	$\underline{\pi}$	$\underline{S}$
0	1	$1+r$
1	3	$S^{(1)}$

with $S^{(1)}(\omega_j) = \begin{cases} 2, & j=1 \\ 3, & j=2 \\ 5, & j=3 \end{cases}$

and r deterministic (i.e. constant)

(a) Find all $r > -1$ such that the market model is A.F.

(b) For the other r give an A.O. $\underline{\xi}$.

(3) (20p) Prove the theorem of the separating hyperplane.

<u>Theorem:</u> Let C be a non-empty convex set in $\mathbb{R}^m$ which does not contain 0.

Then there exist an affine hyperplane H with equation $H = \{ x \in \mathbb{R}^m \mid x \xi = 0 \}$ with $\xi \in \mathbb{R}^m \setminus \{0\}$, such that

- $C \subseteq \{ x \in \mathbb{R}^m \mid \xi x \geq 0 \}$

  and

- $\exists\, z \in C : \quad \xi z > 0.$

[I give some hints in the lecture]

Restrict to the linear span of $\mathcal{C}$,
so you can assume that $\mathcal{C}$ spans $\mathbb{R}^m$

① Prove the case $0 \notin \overline{\mathcal{C}}$ (closure of $\mathcal{C}$) first

③ Then show that $\exists p \in \mathcal{C}\ \exists_Q \in \mathbb{R}^d$

- $0 \in]Q, P[$
- $[Q, P] \cap \mathcal{C} \subseteq]0, P]$

④ Now consider $Q_n := \frac{1}{n} Q + (1 - \frac{1}{n}) \cdot 0$
($Q_n \xrightarrow{n \to \infty} 0$) and apply ②.

Problem Sheet 3 (Mathematical finance I)

1.3.22

Problem 1: (10 points)

We consider a market model $(\underline{\Pi}, \underline{S}, P)$ with

- $\Omega = \{\omega_1, \omega_2\}$, $P(\{\omega_1\}) = \frac{1}{2}$
- $\underline{\Pi} = (1, \Pi^{(1)}, \Pi^{(2)})$, $\Pi^{(1)} = 10$, $\Pi^{(2)} = ?$
- $\underline{S} = (S_0 = 1, S^{(1)}, S^{(2)})$

with $S^{(1)}(\omega) = \begin{cases} 11, & \omega = \omega^+ \\ 9, & \omega = \omega^- \end{cases}$

$S^{(2)}(\omega) = \begin{cases} 23, & \omega = \omega^+ \\ 20, & \omega = \omega^- \end{cases}$

Find the set of all AF prices of

$C = (S^{(1)} + S^{(2)} - 31)^+$.

Problem 2: (10 points) Let (Π, S, P) be an A.F. market model. Show the put-call parity for the prices without using martingal measures, i.e. just by producing arbitrage opportunities.

Problem 3: (15 points). Given an AF market model consisting also of a stock S.

Consider call option $C_K := (S-K)^+$.

Let $K_2 \geq K_1$, $K_i \in \mathbb{R}$.

Show that the prices of C_{K_2}, C_{K_1} must satisfy

(i) $\Pi(C_{K_1}) \geq \Pi(C_{K_2})$

(ii) $\frac{1}{1+r}(K_2 - K_1) \geq \Pi(C_{K_1}) - \Pi(C_{K_2})$

(for r deterministic.)

(iii) $\forall \lambda \in [0,1]$: $\lambda \Pi(C_{K_1}) + (1-\lambda)\Pi(C_{K_2}) \geq \Pi(C_{\lambda K_1 + (1-\lambda)K_2})$

Problem 4 (10 points):

Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose there are μ-atoms $A_1, \dots, A_m$

s.t. $X = A_1 \dot\cup A_2 \dot\cup \dots \dot\cup A_m$.

Then $\dim_{\mathbb{R}} L^\infty(\mathcal{M}, \mu) = m$

Problem Sheet 4 (40 points) 8.3.22

Lecture: Financial mathematics I.

Problem 1 (10 pts) Consider the probability space given by $\left(\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\},\ \mathcal{F} = \mathcal{P}(\Omega),\ P \text{ with } P(\{\omega_i\}) = \frac{1}{2^i},\ i \in \mathbb{N}\right)$

Let $\mathcal{F}_0$ be the sub-σ-algebra of $\mathcal{F}$ generated by the sets $\{\omega_{2i}, \omega_{2i-1}\}$, $i \in \mathbb{N}$.

(a) Compute $E_P[X \mid \mathcal{F}_0]$ for

$$X : \Omega \to \mathbb{R},\ X(\omega_i) := i^3$$

(b) $Q : \mathcal{F} \to [0,1]$ is defined via

$$Q(\{\omega_j\}) = \begin{cases} \frac{1}{2^j}, & \text{if } j \text{ is a power of } 2. \\ 0, & \text{else.} \end{cases}$$

Compute $E_Q[X \mid \mathcal{F}_0]$ for X given in a).

Problem 2: (Fatou's lemma for conditional expectations) (10 pts.)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and

$0 \le X_1 \le X_2 \le X_3 \le \ldots$ a sequence of

$\mathcal{F}$-measurable random variables $X_i : \Omega \to [0,\infty]$.

Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$.

(a) Prove $\quad E[X_1 | \mathcal{F}_0] \le E[X_2 | \mathcal{F}_0]$

(b) Prove $\quad E[\lim_{n\to\infty} X_i | \mathcal{F}_0] = \lim_{n\to\infty} E[X_i | \mathcal{F}_0]$

Problem 3 (10 pts) (A F, no martingale measure in an infinite market model)

· Find an infinite market model $(\underline{\Pi}, \underline{S}, P)$, i.e. with ∞-many assets; $S^{(0)}, S^{(1)}, S^{(2)}, S^{(3)}, \dots$; which is arbitrage free and has no martingale measure.

⌜The definition of an arbitrage opportunity is the following.

$\underline{\xi} = (\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, \dots)$ is called an arbitrage opportunity for $(\underline{\Pi}, \underline{S}, P)$ if

$\sum_{i=0}^{\infty} |\xi^{(i)} \Pi^{(i)}| < \infty$ and

$\sum_{i=0}^{\infty} |\xi^{(i)} S^{(i)}| < \infty$ P-almost surely.

and · $\underline{\xi}\,\underline{\Pi} \leq 0$

· $\underline{\xi}\,\underline{S} \geq 0$ P-almost surely

· $P(\underline{\xi}\,\underline{S} > 0) > 0$. ⌟

—4—

Problem 4) We consider the following market model with 3 timepoints ($t_1 < t_2 < t_3$)

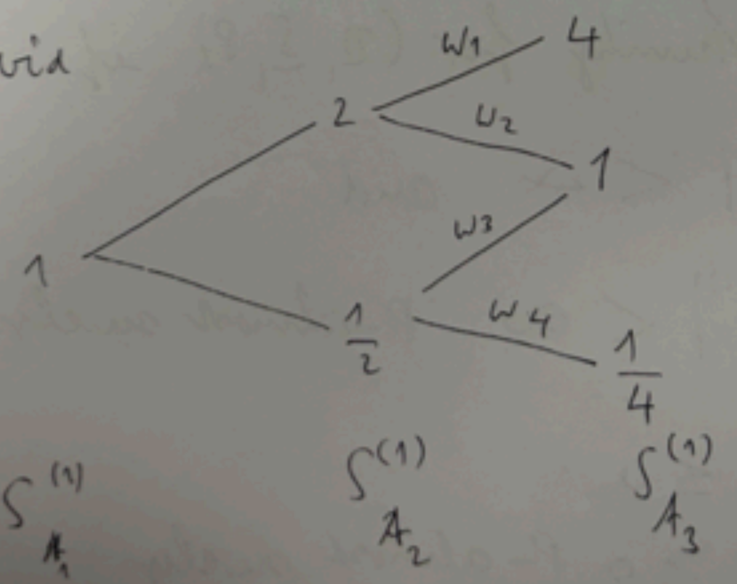
- the bond has vanishing return for t_2 and t_3, and price 1 at t_1.
- we have one asset $S^{(1)}_*$

$$S^{(1)}_* = \Big(\underset{\pi^{(1)}}{\underset{\parallel}{S^{(1)}_{t_1}}}, S^{(1)}_{t_2}, S^{(1)}_{t_3}\Big)$$

defined on $\big(\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$,

$\mathcal{F} := \mathcal{P}(\Omega)$, P with $P(\{\omega_i\}) = \frac{1}{4}\big)$

via



such that

$S^{(1)}_{t_i}$ is $\mathcal{F}_{t_i}$-measurable

$\mathcal{F}_{t_1} := \{\emptyset, \Omega\}$

$\mathcal{F}_{t_2} := \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$

$\mathcal{F}_{t_3} := \mathcal{P}(\Omega)$.

Find the price of a straddle

$C := |S_{t_3} - S_{t_2}|$ for

(a) the time $t = t_1$

(b) — " — $t = t_2$. (Caution: This is a random variable!)

Problem Sheet 5. (20 + 30* pts.)

"Financial mathematics I"

P1) (10pt) Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and Q is a probability measure on $\mathcal{F}$ such that $P << Q$. Let $X \in L^1(\mathcal{F}, P)$ be given and $\mathcal{F}^0 \subseteq \mathcal{F}$ be a σ-subalgebra of $\mathcal{F}$.

Prove that

$$E_P[X \mid \mathcal{F}^0] = \frac{E_Q[X \frac{dP}{dQ} \mid \mathcal{F}^0]}{E_Q[\frac{dP}{dQ} \mid \mathcal{F}^0]}$$

Remark: This is a generalization of the measure transformation formula, see the case $\mathcal{F}^0 = \{\emptyset, \Omega\}$.

P2) Prove the linearity of conditional expectation, i.e. given $(\Omega, \mathcal{F}_0 \subseteq \mathcal{F}, P)$ a probability space with σ-subalgebra $\mathcal{F}_0$.

Then we have for all $\alpha, \beta \in L^\infty(\mathcal{F}_0, P)$ and for all $X, Y \in L^1(\mathcal{F}, P)$ the equality:

$$\alpha E[X|\mathcal{F}_0] + \beta E[Y|\mathcal{F}_0] = E[\alpha X + \beta Y|\mathcal{F}_0]$$

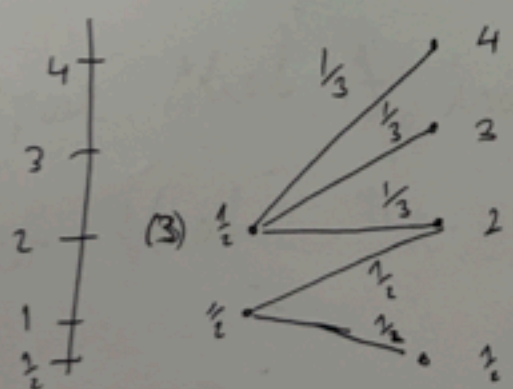
(10 pk.)

Problem 3) (20 points)

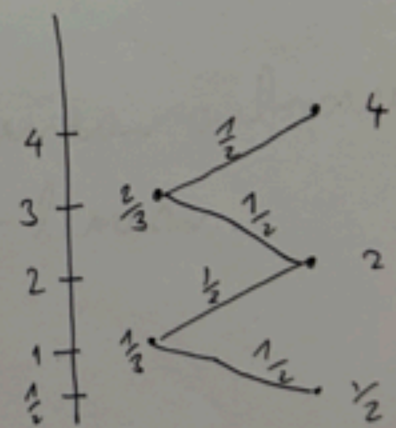
Suppose we are given an AF market model $(\underline{S}_* = (S^{(0)}_*, S^{(1)}_*, S^{(2)}_*), \mathbb{P})$ on $(\Omega, \mathcal{F}_0 \subseteq \mathcal{F}, \mathbb{P})$ such that

- $S^{(0)}_0 = S^{(0)}_1 \equiv 1$
- $S^{(1)}_*$ and $S^{(2)}_*$ are independent w.r.t. $\mathbb{P}$ and we only consider martingale measures such that $S^{(1)}_*$ and $S^{(2)}_*$ are independent.

- $S^{(1)}$



$S^{(2)}$



Find the best superhedging strategy for $(S^{(2)}_1 - S^{(1)}_1)^+$.

Problem 4) (10*pts)

Suppose we are given a market model $(\underline{S}_*, P)$ on $(\Omega, \mathcal{F}_0 \subseteq \mathcal{F}_1, P)$ which is not AF, see Ref. 39.

Show that there is a portfolio $\underline{\xi}$ with bounded $\xi^{(i)}$, $i = 0, .., d$, which is an AO.

Problem sheet 6. (30+20* pts.)

Financial Mathematics I

<u>Problem</u> 1) (10) We have defined

The map $\pi_A : H \longrightarrow A$

for a convex, closed non-empty

subset of a reel Hilbert space H, i.e.

π_A is defined via

$$\|x - \pi_A(x)\|_2 = \inf\{\|x-y\|_2 \mid y \in A\}$$

Suppose that A is a $\mathbb{R}$-linear subspace

Show that π_A is $\mathbb{R}$-linear.

Problem 2) (10) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_n, X \in \mathcal{L}^0(\Omega, \mathcal{F}, P)$, $n \in \mathbb{N}$.
Suppose $X_n \longrightarrow X$ P-a.s.
Show that $X_n \xrightarrow{P} X$ ("X_n converges to X in probability"),
i.e. $\forall \varepsilon > 0$: $P(|X_n - X| \geq \varepsilon) \xrightarrow{n \to \infty} 0$.

Problem 3)* (10*) Show that $L^1(\Omega, \mathcal{F}, P)$ is complete w.r.t. $\|\cdot\|_1$.

Hint: ① Given $X_n \in L^1$ a Cauchy sequence w.r.t. $\|\cdot\|_1$, show that $\exists$ subsequence $(X_{n_j})_{j \in \mathbb{N}}$ which converges P-a.s.:
$X_{n_j} \xrightarrow{j \to \infty} X \in L^0$

② Show that $X \in L^1$ (using Fatou's lemma. Learn about its proof.)

(3) Show that there is a subsequence of $(X_{n_j})_{j\in\mathbb{N}}$ which converges in L^1 to X. (You could use Lebesgue's convergence theorem. Learn about its proof.)

(4) From (3) follows that $X_n \xrightarrow{L^1} X$, because $(X_n)_{\mathbb{N}}$ is a Cauchy ~~...~~ sequence.

—4—

Problem 4) Consider the following multiperiod market model.

Let $U_1, \dots, U_T$ be $N(0, \frac{1}{T})$ i.i.d (independent and identically distributed) variables. ($N(\mu, \sigma^2)$ is the normal distribution with expectation μ and variance σ^2)

For the bond we have $S_t^{(0)} \equiv 1$, $t = \overline{0, T}$.

$(\Omega, \mathcal{F}_t := \sigma(U_1, \dots, U_t), t = 0, \dots, T, P)$

We have one risky asset $S_*^{(1)}$ which we just call S_*.

Suppose $S_T = e^{\sigma W_T}$ with $\sigma > 0$

and $W_T := U_1 + \dots + U_T$

(a)(10) Suppose that P is martingale measure for S_*. ~~St~~.

Compute S_t, $t = 0, \dots, T-1$.

(b)*(10*) Suppose we have a utility function:

$u(x) = \sqrt{x}$.

Find a martingale measure Q for the process $u(S_t)$, $t=\overline{0,T}$ such that $Q \approx P$.

Hint: (a) Maybe it is more convinient to work with

$$B_t = \sqrt{T} \cdot (U_1 + \dots + U_t) \quad , t=\overline{0,T}$$

(b) Find a stochastic process $(F_t)_{t=\overline{0,T}}$ adapted to $\mathcal{F}_*$ such that $(F_t)_t$ is a P-martingal and $(F_t \cdot u(S_t))_t$ is a P-martingale.

Put $\frac{dQ}{dP} = F_T$

Financial Mathematics I

Problem Sheet 7:

Problem 1: (10 points) (self-financing property)

Prove Proposition 58.: Given a MM $(\underline{S}_*, \mathcal{F}_* \subseteq \mathcal{F}, P)$ "let $\underline{\xi}$ be a trading strategy. Then are equivalent:

1° $\underline{\xi}$ is self-financing

2° $\forall_{t=\overline{1,T-1}}$: $\underline{\xi}_t \underline{X}_t = \underline{\xi}_{t+1} \underline{X}_t$

3° $\forall_{t=\overline{1,T}}$: $V_t = V_0 + G_t$

Problem 2: (20 points) (different trading strategies)

Let $(\underline{S}_*, \mathcal{F}_* \subseteq \mathcal{F}, P)$ be a MM

Write down the following trading strategies as a process. Suppose P is trivial on $\mathcal{F}_0$.

(a) At time 0 you invest all together 0€, 1000 € into the 1st asset

— 2 — and nothing into the other risky assets.

You sell all shares of the 1^{st} asset when it reaches a value $> a > S_0^{(1)}$.

(a is fixed here.)

and you then invest the obtained payoff into the bond.

(b) Let $0 < a < b$.

At $\underline{t=0:}$ you invest 1000 € all together. · If $S_0^1 < a$, then you invest 1000 € in the 1^{st} ~~asset~~ asset.

· If $S_0^1 \geq a$, then you invest everything into the bond.

At $\underline{t>0:}$ You invest all shares of the bond into the 1^{st} asset if the 1^{st} asset has down-crossed a, i.e. $S_{t-1}^1 \geq a > S_t^1$.

You sell all shares of the 1^{st} asset if it has up-crossed b, i.e.

$S^1_{t-1} \leq b < S^1_t$, and the payout will completely be invested into the bond.

(c) $0 < a < b$.

At $t = 0$: You invest 1000€.

- You invest all of the 1000 € into the risky assets with value smaller a and in equal terms of money, i.e. the invested value into those assets has to be the same for every asset.

- If all risky assets have a value $\geq a$ then you buy bonds for 1000€

At $t>0$: You sell all riskey assets which up-cross β and use the payout and shares of the bond to buy all riskey assets which down-crossed α in equal terms of money.

Problem 3) (10 points) (martingale property of the value and the gain process)

Let $(\underline{S}_{*}, \mathbb{F}_{*} \subseteq \mathcal{F}, P)$ be a MM.

Suppose $X = (X_t)_{t=\overline{0,T}}$ is a Q-martingale for some probability measure $Q \approx P$. Suppose P is trivial on $\mathcal{F}_0$.

Let $\underline{\xi} = (\underline{\xi}_t)_{t=\overline{1,T}}$ be a self-financing trading strategy such that

$-\xi_t \in L^\infty(\Omega, \mathcal{F}_{t-1}, \mathbb{R}^d)$ for all $t \in \{1, \dots, T\}$

Prove that the value process $V = (V_t)_{t=\overline{0,T}}$ and the gain process are Q-martingales w.r.t. $\mathcal{F}_t$.

Financial Mathematics I (Spring 2022) -1-

Problem Sheet 8 (40 + 20* pts.)

Problem 1) (10 pts) Let X be a real valued stochastic process adapted to $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_T$ a filtration in a σ-algebra $\mathcal{F}$ for Ω.

Let P be a martingale measure for $(X_t)_{t=\overline{1,T}}$ and $Q \approx P$ be a martingale measure for X_0, X_1.

Prove that X is a martingale w.r.t. the measure P^* on $\mathcal{F}$ given by

$$\frac{dP^*}{dP} := E_P\left[\frac{dQ}{dP} \,\Big|\, \mathcal{F}_1\right].$$

Problem 2) (10 pts) Let $(M_t)_{t=\overline{0,T}}$ be a real valued Q-martingale w.r.t. to a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_T$.

Let $s_1 < s_2 < \dots < s_m$ be elements of $\{0, \dots, T\}$.

Prove that:

$$M_{s_2} - M_{s_1},\ M_{s_3} - M_{s_2}, \dots, M_{s_m} - M_{s_{m-1}}$$

are uncorrelated, if M is L^{m-1} ~~square~~ integrable, i.e. $E_Q[M_t^{m-1}] < \infty$

for $0, \dots, T$.

Problem 3) (10 pts.) Let $(M_t)_{t=\overline{0,T}}$ be an integrable martingale w.r.t. $(\Omega, \mathcal{F}_t \text{ in } \mathcal{F}, P)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Suppose $(e^{M_t})_{t=\overline{0,T}}$ is a P-martingale too.

Show that $M_t = M_0$ P-as for all $t = \overline{1,T}$.

<u>Problem 4)</u> (10pts) Given $(\Omega, \mathcal{F}_t \text{ in } \mathcal{F}, P)$, $\mathcal{F}_t = (\mathcal{F}_t)_{t=\overline{0,T}}$ a filtration in $\mathcal{F}$, and a square integrable martingale $(M_t)_{t=\overline{0,T}}$.

<u>Show that</u>

$$N_t := M_t^2 - \sum_{s=0}^{t-1} (M_{s+1} - M_s)^2$$ is a P-martingale w.r.t. $\mathcal{F}_t$.

<u>Problem 5)*</u> (20pts) Let $U_1, \dots, U_T$ be i.i.d $N(0,1)$ distributed variables and consider $\mathcal{F}_t := \sigma(U_1, \dots, U_t)$ for $t = \overline{0,T}$.

We consider two assets:

$$B_t := e^{rt}, \quad t = \overline{0,T} \quad \text{(bond)}$$

$$S_t := S_0 \, e^{\left(\sigma(U_1 + \dots + U_t) - \frac{\sigma^2}{2} t + \alpha t\right)},$$

$t = \overline{0,T}$ and $S_0 \in \mathbb{R}^{>0}$ and $\sigma > 0$.

—4—

Find an AF price for a call option

$C^{call} := (S_T^{\cdot} - K)^{+}$.

— Financial Mathematics I

Problem Sheet 9

Problem 1: (10pts) Let $X \in L^0(\Omega, \mathcal{F}, P)$ with two values $a \neq b$, i.e. we have $\mathrm{im}(X) = \{a, b\}$ and $P(X=a) \in]0,1[$. Let $\mathcal{U} \subseteq \mathcal{F}$ be a sub-σ-algebra such that $E[X|\mathcal{U}]$ is constant.

(a) Show that X is independent to $\mathcal{U}$.

(b) What happens if we allow X to take 3 different values with positive probability?

Problem 2: (10 pts)

Prove Proposition 5.3. in the text book.

Prop 5.3.: Let $(U_t)_{t=\overline{0,T}}$ be an adapted process w.r.t. $(\Omega, (\mathcal{F}_t)_{t=\overline{0,T}}, Q)$ and suppose $U_t \in L^1(\Omega, \mathcal{F}_t, Q)$ for all $t = \overline{0,T}$.

Then are equivalent

1° U is a Q-supermartingale

2° $-U$ is a Q-submartingale

3° $\forall_{0 \le s < t \le T}$: $U_s \ge E[U_t \mid \tilde{\mathcal{F}}_s]$

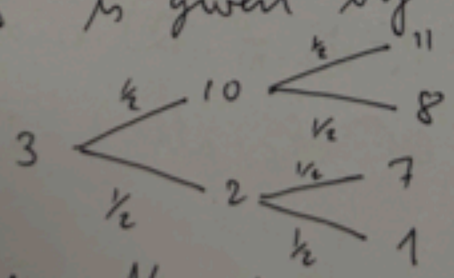
4° $\forall_{1 \le t \le T}$: $U_{t-1} \ge E[U_t \mid \mathcal{F}_{t-1}]$.

Problem 3) (10 pts)

Consider a market model with 2 periods, a zero-bond (no return, i.e. $S_0^{(0)} = S_1^{(0)} = S_2^{(0)} \equiv 1$) and one risky asset $S^{(1)}$, i.e.

$(\underline{S}, \mathcal{F}_t, P)$ such that

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(S_1^{(1)})$,
  $\mathcal{F}_2 = \sigma(S_1^{(1)}, S_2^{(1)})$.
- $S^{(1)}$ is given by



where the edges are marked by

The conditional probabilities w.r.t. P

(i.e. $P(S_2^{(1)} = 8 \mid S_1^{(1)} = 10) = \frac{1}{2}$)

Compute the Snell envelope for the American call option

$C = (C_0, C_1, C_2)$ with

$$C_t = (S_t^{(1)} - 4)^+, \; t = 0, 1, 2.$$