LINEAR ALGEBRA 1
PROBLEM SHEET 7

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Problem 1 (8 + 10 + 10*, warm up for vector spaces). (i) Let $(\mathrm{V}, +, \cdot)$ be a vector space. Prove
  (a) that there is only one neutral element,
  (b) that for every $v \in \mathrm{V}$ the additive inverse is uniquely determined (we denote it by $-v$)
  (c) and show:
$$-(-v) = v \text{ and } 0_{\mathbb{R}} \cdot v = 0_{\mathrm{V}}.$$
(ii) Show that the "strange" example, see Example 153(e), is a vector space.
(iii) (*) Think about why the "strange" example is in fact not so strange.

Problem 2 (5+5+10, subspaces). (i) Prove that the intersection of two subspaces of a vector space is a subspace.
(ii) Let W, U be subspaces of a vector space V. We define their sum via
$$\mathrm{W} + \mathrm{U} := \{w + u \mid \; w \in \mathrm{W}, \; u \in \mathrm{U}\}.$$
Prove that $\mathrm{W} + \mathrm{U}$ is a subspaces of V.
(iii) Find all subspaces of $\mathbb{R}^3$ which contain the line $\mathrm{L}_{(1,1,0)}$ (this is the line through $(1, 1, 0)$ and the origin.)

Problem 3 (20, Linear independence). Which of those families of vectors are linearly independent?
(i) $(1, -2, 3, 1), (1, 1, 1, 1), (1, 0, 0, 1)$ in $\mathbb{R}^4$
(ii) $(1, 2, 1, -1), (2, 1, 0, 0), (1, 2, 1, 1), (0, -1.5, -1, -3)$ in $\mathbb{R}^4$
(iii) In $\mathrm{Map}(\mathbb{R}, \mathbb{R})$: $1, \cos(x), \cos(2x), \cos(3x)$
(iv) In $\mathrm{Map}(\mathbb{R}, \mathbb{R})$: $1, \cos(x), \cos(2x), \cos(3x), \cos^3(x)$

Problem 4 (10 + 10 + 10* + 20*,cross product and generalisation). (i) Compute the following cross products
$$(1, 3, 5) \times (-1, 2, 6), \; (-2, 1, -1) \times (1, 1, -1)$$
(ii) Find a normal for the hyperplane through the points $(1, 1, 3), (1, -1, 2), (1, 1, 1)$ whose length is the area of the triangle with those points as vertexes.
(iii) (*) Try to generalise the cross product to higher dimensions, i.e. define a suitable vector product $\times(v^{(1)}, v^{(2)}, \ldots, v^{(n-1)}) \in \mathbb{R}^n$ for vectors $v^{(1)}, \ldots, v^{(n-1)} \in \mathbb{R}^n$. Give a formula and describe the length and the direction.
(iv) (*) Most ambitious: Give a proof for your assertions in (iii).

Date: Please hand in before the lecture by **22th of November 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.