Chapter III

Euclidean vector spaces

$n \in \mathbb{N}$.

III.1. Vectors

Def 102. The set of ordered n-tuples of real numbers is called n-space and denoted by $\mathbb{R}^n$, i.e.

$$\mathbb{R}^n = \{ \underbrace{(x_1, \dots, x_n)}_{\text{coordinates of the point}} \mid x_1, \dots, x_n \in \mathbb{R} \}$$

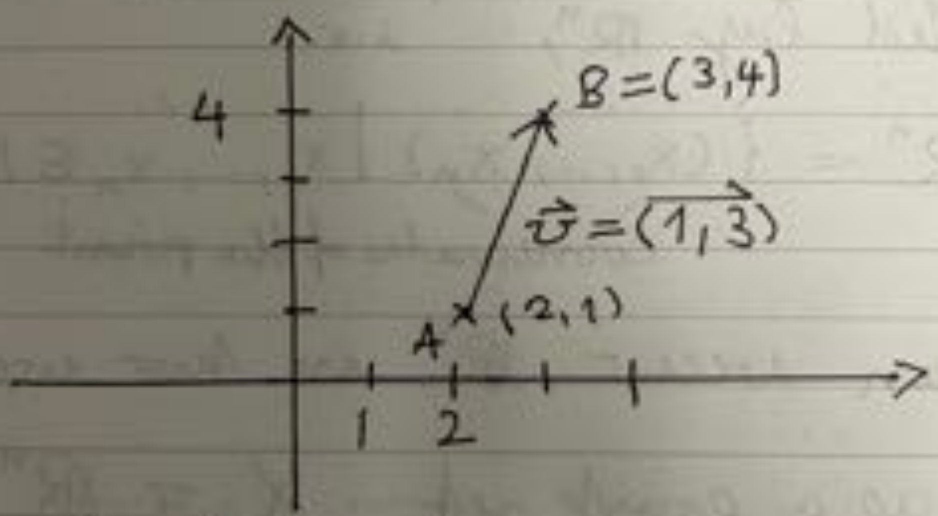
We can view $\mathbb{R}^n$ in two ways.

- as a point set $X := \mathbb{R}^n$. We will write capital letters for points: $A = (a_1, \dots, a_n) \in \mathbb{R}^n$.
- as a set of vectors $V := \mathbb{R}^n$; $\vec{v} = (v_1, \dots, v_n) \in V$. $v_1, \dots, v_n$ are called the components of $\vec{v}$.

We can map a point $A \in X$ to another point using a vector $\vec{v} \in V$; "We <u>translate</u> A <u>along $\vec{v}$</u>"

$$t_{\vec{v}} : X \longrightarrow X$$

$$t_{\vec{v}}(A) :=: B = (a_1 + v_1, \ldots, a_n + v_n)$$



$$t_{(1,3)}(A) = B$$

We can represent a vector by a directed segment, which consists of

- an "initial point" A
- a "terminal point" B
- and the vector

$$\overrightarrow{AB} = (b_1 - a_1, \ldots, b_n - a_n)$$

Notation 103: Sometimes we write a vector using a column

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Remark 104: Points are just points. Vectors are used to translate points. So we need a <u>length</u> and a <u>direction</u>.

Def 104: Given $\vec{v} = (v_1, \dots, v_n) \in V = \mathbb{R}^n$, the value

$$\|\vec{v}\| := \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

is called the <u>norm</u> (length, magnitude) of $\vec{v}$.

Example 105: (a) $\vec{v} = (1, 3) \in \mathbb{R}^2$

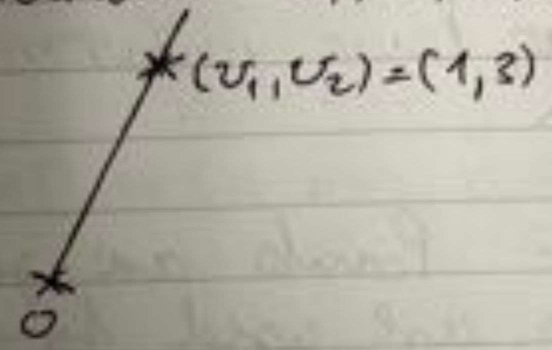
$$\|\vec{v}\| = \sqrt{1+9} = \sqrt{10}.$$

$$\|(2, -1, 3, 4, 0, 1)\| = (4+1+9+16+1)^{1/2} = \sqrt{31}$$

for $\vec{w} = (2, -1, 3, 4, 0, 1) \in \mathbb{R}^6$.

114+3

The direction is given by the ray starting at $0 \in \mathbb{R}^n$ going through the point with coordinates $v_1, \ldots, v_n$.

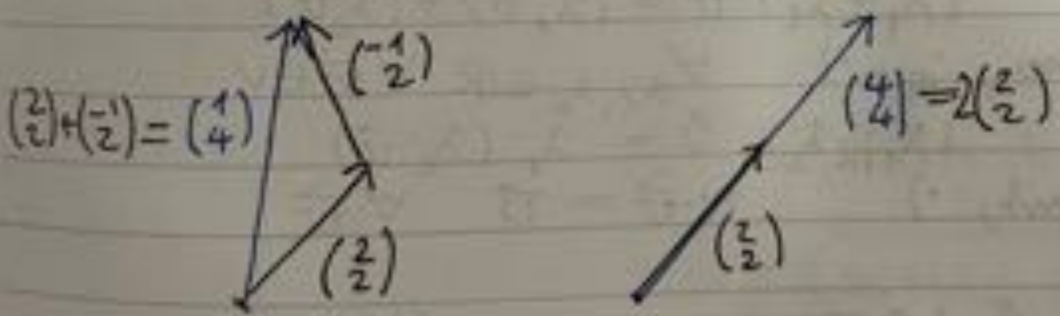


III 2. Vector space structures on $V = \mathbb{R}^n$

We have two structures on V:

(1) $+ : V \times V \longrightarrow V$ (an addition) defined via
$$\vec{v} + \vec{w} := \overrightarrow{(v_1 + w_1, v_2 + w_2, \dots, v_n + w_n)}$$
and

(2) $\cdot : \mathbb{R} \times V \longrightarrow V$ (a scalar multiplication) via
$$\lambda \cdot \vec{v} := \overrightarrow{(\lambda v_1, \dots, \lambda v_n)}$$



We recall

<u>Prop 106</u>: $(V, +, \cdot)$ satisfies the following properties:

(1) (com +) $\vec{v} + \vec{w} = \vec{w} + \vec{v} \quad \forall \vec{v}, \vec{w} \in V$
"commutativity of +"

(ass +) $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u}) \quad \forall \vec{u}, \vec{v}, \vec{w} \in V$
"associativity of +"

(~~neu~~ 0 +) The element $\vec{0} = (0, \dots, 0)$ satisfies $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v} \ \forall \vec{v} \in V$

~~[illegible]~~ "existence of a neutral element"

(inv +) For every $\vec{v} \in V$ there exists a $\vec{w} \in V$ such that $\vec{v} + \vec{w} = \vec{w} + \vec{v} = \vec{0}$.

"existence of an inverse"

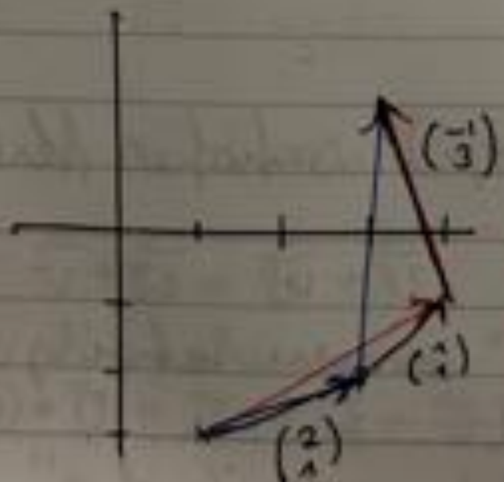
(2) (ldist, ·, +) $\forall \lambda \in \mathbb{R} \ \forall \vec{v}, \vec{w} \in V$:
$$\lambda(\vec{v} + \vec{w}) = (\lambda \vec{v}) + (\lambda \vec{w}).$$

(rdist, $+_{\mathbb{R}}$, ·) $\forall \lambda_1, \lambda_2 \in \mathbb{R} \ \forall \vec{v} \in V$:
$$(\lambda_1 + \lambda_2)\vec{v} = (\lambda_1 \vec{v}) + (\lambda_2 \vec{v})$$

(ass, $\cdot_{\mathbb{R}}$, ·) $\forall \lambda_1, \lambda_2 \in \mathbb{R} \ \forall \vec{v} \in V$
$$(\lambda_1 \cdot_{\mathbb{R}} \lambda_2) \cdot \vec{v} = \lambda_1 \cdot (\lambda_2 \cdot \vec{v})$$

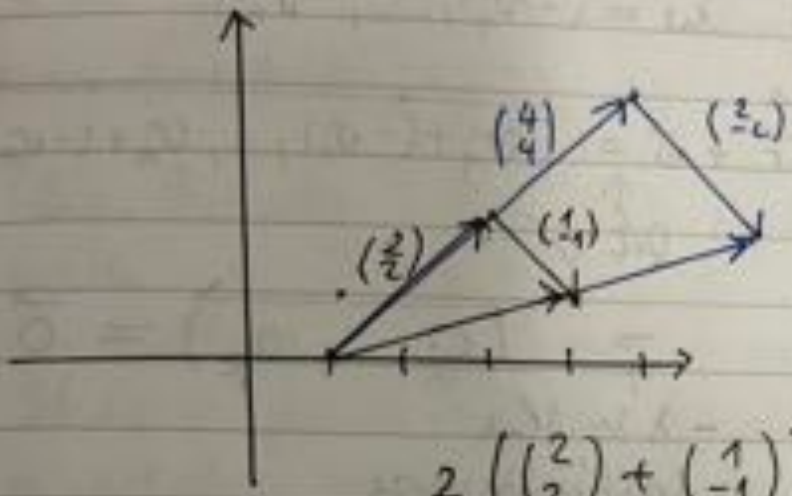
(unit, ·) $1 \cdot \vec{v} = \vec{v} \quad \forall \vec{v}$

Example 107: (a) For (ass, $+_V$):



$$\binom{2}{1} + \Big(\overbrace{\binom{1}{1} + \binom{-1}{3}}^{\binom{0}{4}}\Big)$$
$$= \underbrace{\Big(\binom{2}{1} + \binom{1}{1}\Big)}_{\binom{3}{2}} + \binom{-1}{3}$$

(b) For (ldist, $\cdot$, $+$) "left distributivity".



$$2\left(\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}\right)=2\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}6\\2\end{pmatrix}$$

$$2\begin{pmatrix}2\\2\end{pmatrix}+2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}4\\4\end{pmatrix}+\begin{pmatrix}2\\-2\end{pmatrix}=\begin{pmatrix}6\\2\end{pmatrix}$$

Proof (Prop 106)

(1) (ass, $+_V$): Take $\vec{v}, \vec{w}, \vec{u} \in V = \mathbb{R}^n$

Then
$$(\vec{v}+\vec{w})+\vec{u} \underset{\text{Def } +_V}{=} \overrightarrow{(v_1+w_1,\dots,v_n+w_n)}+\vec{u}$$
$$\underset{\text{Def } +_V}{=} \overrightarrow{((v_1 +_{\mathbb{R}} w_1) +_{\mathbb{R}} u_1, \dots, (v_n +_{\mathbb{R}} w_n) +_{\mathbb{R}} u_n)}$$
$$\underset{+_{\mathbb{R}} \text{ is associative}}{=} \overrightarrow{(v_1 +_{\mathbb{R}} (w_1 +_{\mathbb{R}} u_1), \dots, v_n +_{\mathbb{R}} (w_n +_{\mathbb{R}} u_n))}$$
$$\underset{\text{Def } +_V}{=} \vec{v} + \overrightarrow{(w_1+u_1,\dots,w_n+u_n)}$$
$$\underset{*}{=} \vec{v} + (\vec{w}+\vec{u}).$$

~~(c)~~ (inv, +) Take $\vec{v} \in V = \mathbb{R}^n$.

Put $\vec{w} = \overrightarrow{(-v_1, \ldots, -v_n)}$.

Then

$$\vec{v} + \vec{w} \underset{\substack{\uparrow \\ \text{Def } +_V}}{=} \overrightarrow{(v_1 + (-v_1), \ldots, v_n + (-v_n))}$$

$$\underset{\substack{\uparrow \\ -\lambda \text{ is the} \\ \text{additive inverse} \\ \text{of } \lambda \text{ in } \mathbb{R}}}{=} \overrightarrow{(0_{\mathbb{R}}, \ldots, 0_{\mathbb{R}})} = \vec{0}$$

Similar for $\vec{w} + \vec{v}$.

The remaining assertions are left as an exercise. $\square$

<u>Remark 108:</u> There is only one additive inverse for $\vec{v}$.

<u>Proof:</u> Let $\vec{w}, \vec{u}$ be additive inverses for $\vec{v}$. Then

$$\vec{u} \underset{\substack{\uparrow \\ (0,+)}}{=} \vec{u} + \vec{0} \underset{\substack{\uparrow \\ \vec{w} \text{ add inv.}}}{=} \vec{u} + (\vec{v} + \vec{w}) \underset{\substack{\uparrow \\ (ass,+)}}{=} (\vec{u} + \vec{v}) + \vec{w}$$

$$\underset{\substack{\uparrow \\ \vec{u} \text{ add inv.}}}{=} \vec{0} + \vec{w} \underset{\substack{\uparrow \\ (0,+)}}{=} \vec{w}$$

$\square$

We write $-\vec{v}$ for the additive inverse of $\vec{v}$, i.e. $-\vec{v} := \overrightarrow{(-v_1, \ldots, -v_n)}$.

<u>Remark 109:</u> 1) The properties in Proposition 106 are called the "vector space ~~properties~~ axioms"

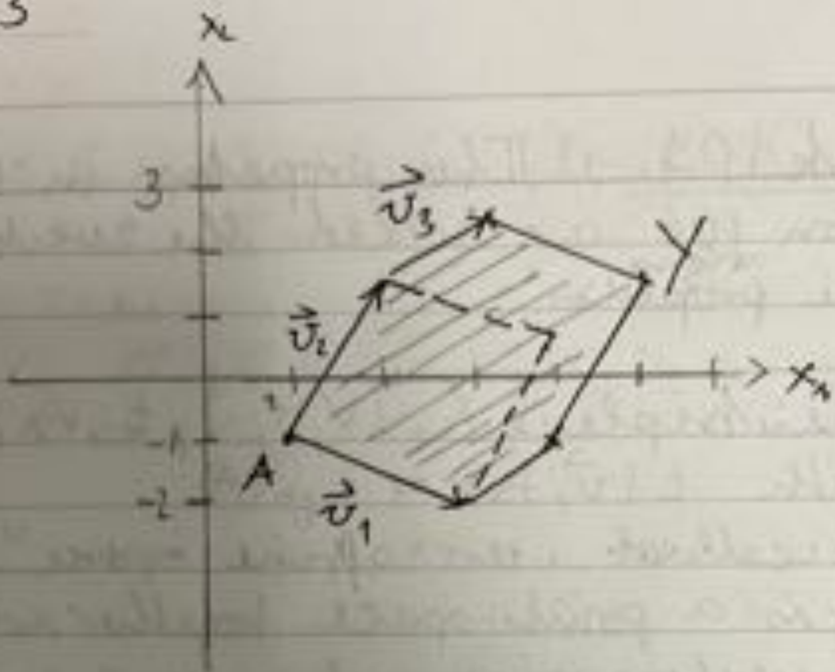
2) The triple $(X, (V, +, \cdot), t: V \times X \longrightarrow X)$ with $t(\vec{v}, A) := t_{\vec{v}}(A)$ is called an "affine space". It is a point space together with a vector space where one can translate points along vectors.

<u>Example 110:</u> (a) $A = (1, -1) \in X = \mathbb{R}^2$

$$\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2{,}5 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Put $Y := \{ t_{(\lambda_1 \vec{v}_1 + \lambda_2 \vec{v} + \lambda_3 \vec{v})}(A) \mid \lambda_1, \lambda_2, \lambda_3 \in [0, 1] \}$

We draw Y.



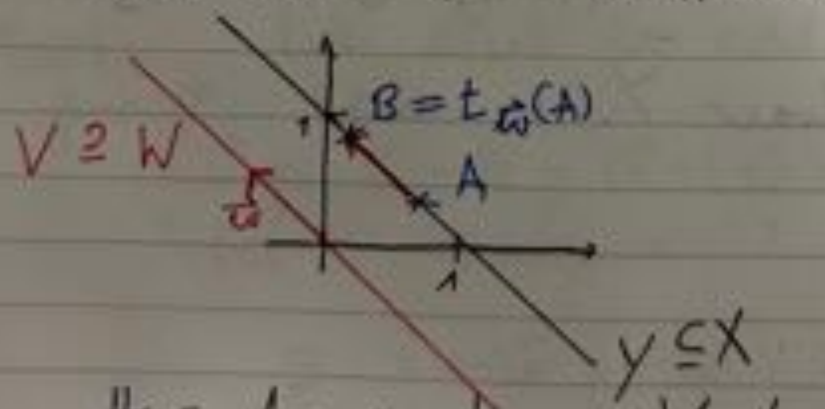
Describe Y using inequalities.

I $\quad 8 \geq x_1 + 2x_2 \geq -1$

II $\quad 11 \geq 2.5x_1 - x_2 \geq 3.5$

III $\quad 5 \geq x_1 - x_2 \geq 0.5$

(b) Consider $(X = \mathbb{R}^2, V = \mathbb{R}^2)$ and the line $Y := \{(x_1, x_2) \in X \mid x_1 + x_2 = 1\}$



How to make Y to an affine space?

Consider $W := \{ \binom{v_1}{v_2} \in V \mid v_1 + v_2 = 0 \}$

1st $(W, +|_{W\times W}, \cdot|_{\mathbb{R}\times W})$ is a vector space, i.e. satisfies the — " — axioms:

For that we just need to show that $+$ and $\cdot$ map into W.

$+$: $\vec{w}^{(1)}, \vec{w}^{(2)} \in W$ To show $\vec{w}^{(1)} + \vec{w}^{(2)} \in W$

$$\vec{w}^{(1)}+\vec{w}^{(2)} = (\underbrace{w_1^{(1)}+w_1^{(2)}}_{=:u_1}, \underbrace{w_2^{(1)}+w_2^{(2)}}_{=:u_2})$$

and

$$u_1+u_2 = (w_1^{(1)}+w_1^{(2)}) + (w_2^{(1)}+w_2^{(2)})$$

$$= (w_1^{(1)}+w_2^{(1)}) + (w_1^{(2)}+w_2^{(2)})$$

$$\underset{\vec{w}^{(1)},\vec{w}^{(2)}\in W}{=} 0$$

$\Rightarrow \vec{u} \in W$

$\cdot$: $\lambda \in \mathbb{R}$, $\vec{w} \in W$

$\lambda \vec{w} = (\lambda w_1, \lambda w_2)$ and

$$\lambda w_1 + \lambda w_2 \underset{(\ell \text{ dist})}{=} \lambda(w_1+w_2) = \lambda \cdot 0_{\mathbb{R}} = 0_{\mathbb{R}}$$

$\Rightarrow \lambda\vec{w} \in W$.

2^{nd} We also need that $t: W \times Y \longrightarrow V$ has its image in V.

$\vec{w} \in W$, $A \in Y$. Put $B := t_{\vec{w}}(A)$

$$\Rightarrow (b_1, b_2) = (a_1 + w_1, a_2 + w_2)$$

and $b_1 + b_2 = a_1 + w_1 + a_2 + w_2$
$= 1$

$\Rightarrow B \in Y$.

(Y, W) is an affine space.

<u>Remark III:</u> For an affine space one needs some more axioms on $t: V \times X \longrightarrow X$.
Have an idea what axioms are needed?

(aff 1) $t_{\vec{0}}(A) = A \quad \forall A \in X$

(aff 2) $t_{(\vec{v}+\vec{w})}(A) = t_{\vec{v}}(t_{\vec{w}}(A))$

(aff 3) $\forall_{A,B \in X} \exists \vec{v} \in V: t_{\vec{v}}(A) = B$.

III 3 The dot product on $\mathbb{R}^n$

We need on $V = \mathbb{R}^n$ an extra structure to be able to
- compute the length of a vector and
- the angle between two non-zero vector.

Def 112: (dot product)

The dot product is the map
$$\bullet : V \times V \longrightarrow \mathbb{R}$$
defined via $\vec{v} \bullet \vec{w} := \sum_{i=1}^{n} v_i w_i$

$$= (v_1, \ldots, v_n) \circ \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$ (using matrix multiplication $\circ$).

"$\bullet$" is also called the Euclidean inner product.

Example 113:

(a) $\vec{v} = \overrightarrow{(1, -1, \tfrac{1}{2}, 3, 5)} \in V = \mathbb{R}^5$

$\vec{w} = \overrightarrow{(0, 2, 7, 13, 1)} \in V$

$$\vec{v} \bullet \vec{w} = 1 \cdot 0 + (-1) \cdot 2 + \tfrac{1}{2} 7 + 3 \cdot 13 + 5 \cdot 1$$
$$= -2 + 3.5 + 39 + 5 = \cancel{47}\ 45{,}5$$

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(b) The dot product can be used to describe $\|\cdot\|$.

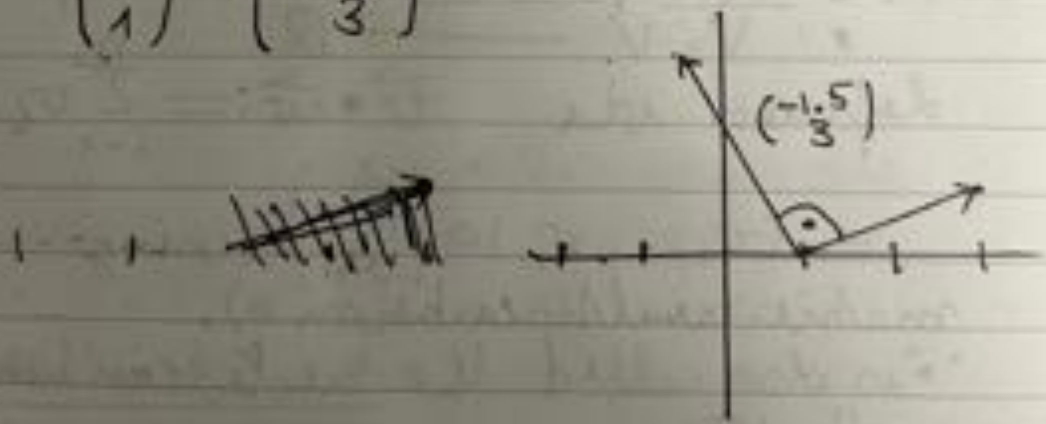
$$\|\vec{v}\| = |\vec{v}\cdot\vec{v}|^{\frac{1}{2}}$$

E.g.: For $\vec{v}$ in (a):

$$\|\vec{v}\| = |\vec{v}\cdot\vec{v}|^{1/2} = \sqrt{1+1+\frac{1}{4}+9+25}$$
$$= \sqrt{36.25}$$

(c) $\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}-1.5\\3\end{pmatrix} \in V = \mathbb{R}^2$

$$\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}-1.5\\3\end{pmatrix} = 2(-1.5) + 1\cdot 3 = 0$$



Def 114: (orthogonal vectors)

Two vectors $\vec{v}, \vec{w} \in V$ are called orthogonal (or perpendicular) if $\vec{v}\cdot\vec{w} = 0$.

Def 115: A vector of norm 1 is called a unit vector.

Ex: (a) $\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\| = \sqrt{1^2+0^2} = 1$

$\begin{pmatrix}1\\0\end{pmatrix}$ is a unit vector

(b) $\left\|\begin{pmatrix}2\\3\\-1\end{pmatrix}\right\| = \sqrt{4+9+1} = \sqrt{14}$

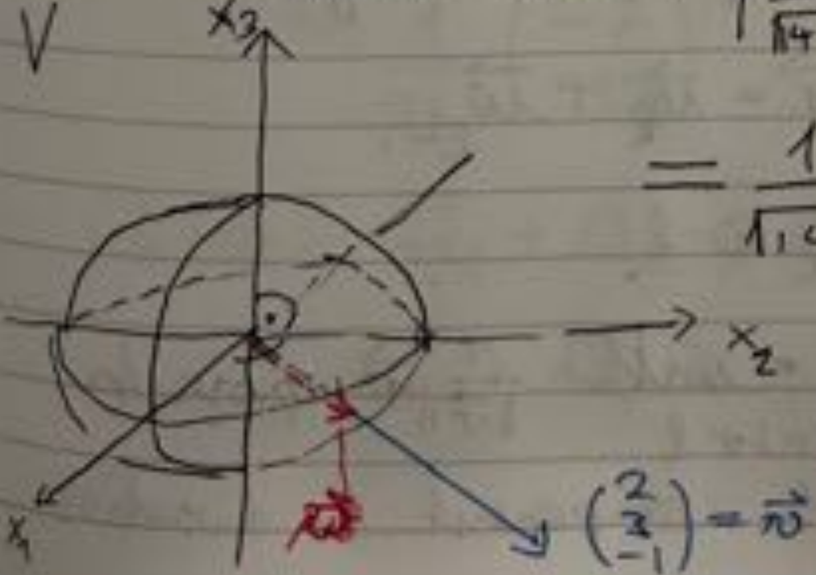
$\vec{v} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$ is not a unit vector

We can "normalize" the vector $\vec{v}$.

$$\vec{v}_{norm} := \vec{w} := \frac{1}{\|\vec{v}\|}\cdot\vec{v} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\-1\end{pmatrix} = \begin{pmatrix}\frac{2}{\sqrt{14}}\\ \frac{3}{\sqrt{14}}\\ -1/\sqrt{14}\end{pmatrix}$$

Then $\|\vec{w}\| = \sqrt{\frac{2^2}{\sqrt{14}^2} + \frac{3^2}{\sqrt{14}^2} + \frac{1^2}{\sqrt{14}^2}}$

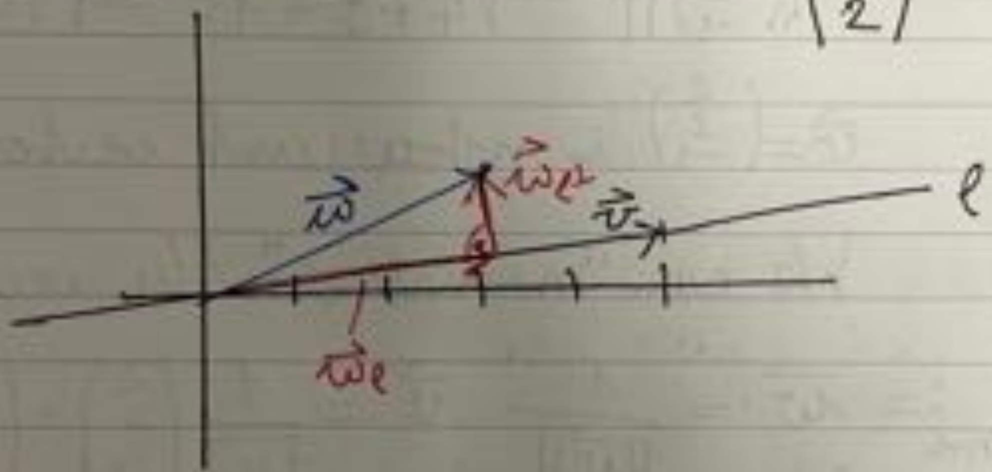
$$= \frac{1}{\sqrt{14}}\,\|\vec{v}\| = \frac{\sqrt{14}}{\sqrt{14}} = 1.$$



$\vec{w}$ lies on the unit sphere.

Remark 116: (geometric interpretation of the dot product)

Consider the line $\ell \subseteq V = \mathbb{R}^2$ through $\vec{v} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and a vector $\vec{w} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$



We want to write $\vec{w}$ as the sum of two orthogonal vectors

$$\vec{w} = \vec{w}_\ell + \vec{w}_{\ell^\perp}$$

such $\vec{w}_\ell \in \ell$.

Idea: The $\cdot$ with $\frac{1}{\|\vec{v}\|}\vec{v}$ projects $\vec{w}$ onto ℓ

$\vec{w} \cdot \vec{v}_{norm}$ is the component in direction of $\vec{v}_{norm}$.

$$\vec{w}_\ell := \left(\vec{w} \cdot \frac{\vec{v}}{\|\vec{v}\|}\right) \frac{\vec{v}}{\|\vec{v}\|} \in \ell$$

$$= \left(\begin{pmatrix}3\\2\end{pmatrix} \cdot \begin{pmatrix}5 \cdot \frac{1}{\sqrt{26}}\\ 1/\sqrt{26}\end{pmatrix}\right) \begin{pmatrix}5/\sqrt{26}\\ 1/\sqrt{26}\end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{26}} 15 + \frac{1}{\sqrt{26}} 2\right) \begin{pmatrix}5\\1\end{pmatrix} \cdot \frac{1}{\sqrt{26}}$$

$$= \frac{17}{26} \begin{pmatrix}5\\1\end{pmatrix} = \begin{pmatrix}85/26\\ 17/26\end{pmatrix}$$

$$\vec{w}_{\ell\perp} := \vec{w} - \vec{w}_\ell$$

$$= \begin{pmatrix}3\\2\end{pmatrix} - \frac{17}{26}\begin{pmatrix}5\\1\end{pmatrix}$$

$$= \frac{1}{26}\begin{pmatrix}-7\\35\end{pmatrix}$$

Check: $\vec{w}_\ell + \vec{w}_{\ell\perp} = \vec{w}$ ✓

$$\vec{w}_\ell \cdot \vec{w}_{\ell\perp} = \frac{17}{26^2}(5(-7) + 1 \cdot 35)$$

$$= \frac{17}{26^2} \cdot 0 = 0. \checkmark$$

This works in general if $\vec{v} \neq \vec{0}$

<u>Prop 117</u>

Let $\vec{w}, \vec{v} \in V = \mathbb{R}^n$. $L := \mathbb{R}\vec{v} \subseteq V$.

Put $\vec{u} = \vec{v}_{norm} := \frac{1}{\|\vec{v}\|} \vec{v}$.

<u>Then</u> (a) $\vec{w} = \vec{w}_L + \vec{w}_{L^\perp}$,

for some $\vec{w}_L \in L$ and $\vec{w}_{L^\perp} \in V$ orthogonal to $\vec{w}_L$.

(b) The decomposition in (a) is unique. In fact we have

$$\vec{w}_L = (\vec{w} \cdot \vec{u})\vec{u} \text{ and}$$
$$\vec{w}_{L^\perp} = \vec{w} - \vec{w}_L$$

<u>Proof:</u> Exercise in example class. $\square$

Example 118: $(n=4)$ $\vec{w} = (1, -1, 2, 1)$
$\vec{v} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$

$\Rightarrow \|\vec{v}\| = 1$, i.e. $\vec{v}_{norm} = \vec{v}$

$\vec{w} = \vec{w}_{\vec{v}} + \vec{w}_{\vec{v}^\perp}$ with

$$\vec{w}_{\vec{v}} = (\vec{w} \cdot \vec{v})\vec{v} = \frac{3}{\sqrt{2}}(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$$
$$= (\frac{3}{2}, 0, \frac{3}{2}, 0)$$

and $\vec{w}_{\vec{v}\perp} = \vec{w} - \vec{w}_{\vec{v}} = (-\frac{1}{2}, -1, \frac{1}{2}, 1)$.

<u>Prop 119</u>: (Properties of the dot product)

The dot product is

(a) bilinear, i.e. for all $\lambda \in \mathbb{R}$ and $\vec{w}, \vec{v}^{(1)}, \vec{v}^{(2)} \in V$ we have

(bi-additivity)

$$(\vec{v}^{(1)} + \vec{v}^{(2)}) \cdot \vec{w} = \vec{v}^{(1)} \cdot \vec{w} + \vec{v}^{(2)} \cdot \vec{w}$$

$$\vec{w} \cdot (\vec{v}^{(1)} + \vec{v}^{(2)}) = \vec{w} \cdot \vec{v}^{(1)} + \vec{w} \cdot \vec{v}^{(2)}$$

(bi-homogeneous property)

$$(\lambda \vec{v}) \cdot \vec{w} = \lambda (\vec{v} \cdot \vec{w})$$

$$\vec{v} \cdot (\lambda \vec{w}) = \lambda (\vec{v} \cdot \vec{w})$$

(b) symmetric: $\forall \vec{v}, \vec{w} \in V$:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

(c) positive definite: $\forall \vec{v} \in V$:

$$\vec{v} \cdot \vec{v} \geq 0$$

and

$$(\vec{v} \cdot \vec{v} = 0 \Leftrightarrow \vec{v} = 0)$$

<u>Proof:</u> (a) Follows from the properties of matrix multiplication as

$$\vec{v} \cdot \vec{w} = (v_1, \dots, v_n) \circ \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

(b) $\vec{v} \cdot \vec{w} = v_1 w_1 + \dots + v_n w_n$

$= w_1 v_1 + \dots + w_n v_n$

$\uparrow$ commutative $\cdot_{\mathbb{R}}$

$= \vec{w} \cdot \vec{v}$

(c) 1) $\vec{v} \cdot \vec{v} = \sum_{i=1}^{n} v_i^2 \geq 0$

2) $\vec{0} \cdot \vec{0} = \sum_{i=1}^{n} 0^2 = 0$

3) Suppose $\vec{v} \cdot \vec{v} = 0$. Then

$\sum_{i=1}^{n} v_i^2 = 0$. Assume $\exists_{i_0 \in \{1,\dots,n\}} : v_{i_0} \neq 0$

Then $\vec{v} \cdot \vec{v} \geq v_{i_0}^2 > 0$ ↯

Thus $\vec{v} = \vec{0}$.

$\square$

(b) $\vec{v} \cdot \vec{w} = v_1 w_1 + \ldots + v_n w_n$

$$= \underset{\substack{\uparrow \\ \cdot_{\mathbb{R}} \text{ commutative}}}{} w_1 v_1 + \ldots + w_n v_n$$

$$= \vec{w} \cdot \vec{v}$$

(c) 1) $\vec{v} \cdot \vec{v} = \sum_{i=1}^{n} v_i^2 \geq 0$

2) $\vec{0} \cdot \vec{0} = \sum_{i=1}^{n} 0^2 = 0$

3) Suppose $\vec{v} \cdot \vec{v} = 0$. Then

$\sum_{i=1}^{n} v_i^2 = 0$. Assume $\exists_{i_0 \in \{1,\ldots,n\}} : v_{i_0} \neq 0$

Then $\vec{v} \cdot \vec{v} \geq v_{i_0}^2 > 0$ $\lightning$

Thus $\vec{v} = \vec{0}$.

$\square$

Example 120 (a) in $V = \mathbb{R}^3$

$$\left(\overrightarrow{(0,1,-1)} + \overrightarrow{(1,2,3)}\right) \cdot \overrightarrow{(7,0,9)} = \overrightarrow{(1,3,2)} \cdot \overrightarrow{(7,0,9)}$$

$$= 7 \cdot 1 + 3 \cdot 0 + 2 \cdot 9 = 7 + 18 = 25$$

$$\overrightarrow{(0,1,-1)} \cdot \overrightarrow{(7,0,9)} + \overrightarrow{(1,2,3)} \cdot \overrightarrow{(7,0,9)}$$

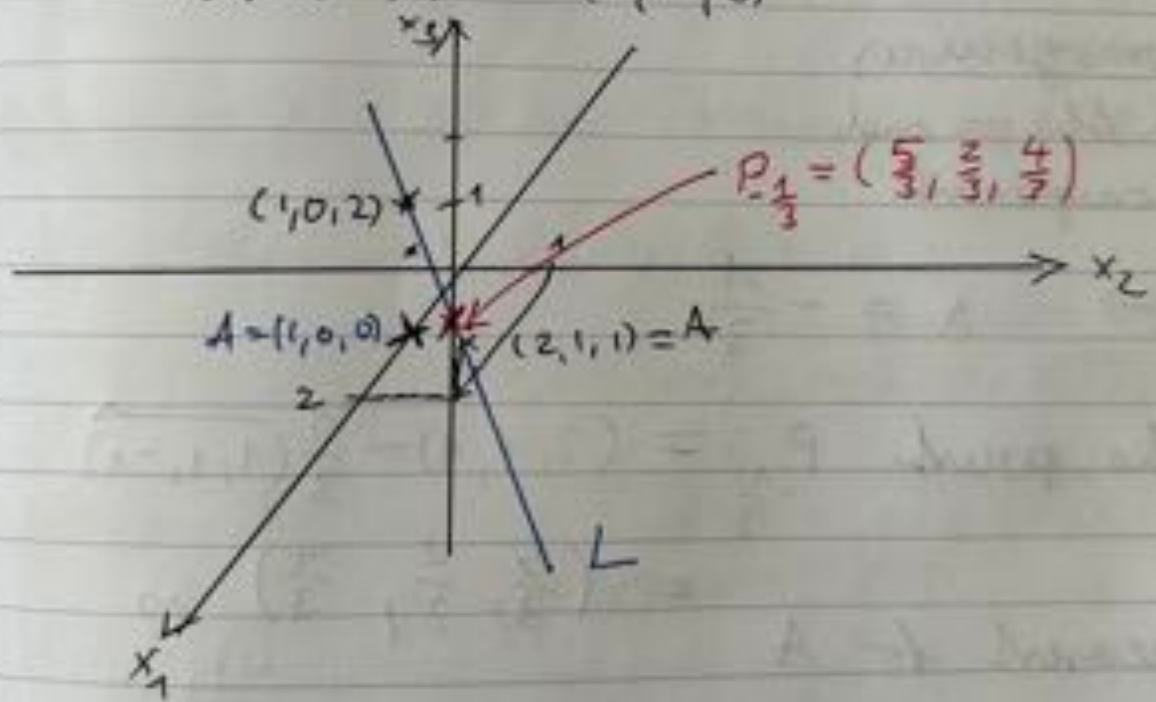
$$= -9 + (7 + 27) = 25$$

(b) We consider the following line in $X = \mathbb{R}^3$.

$$L := (2,1,1) + \mathbb{R}\overrightarrow{(1,1,-1)} \subseteq X$$

"we also write + for $t_{\overrightarrow{(1,1,-1)}}(2,1,1)$"

Take $A = (1,0,0)$



Which point on L is nearest to A? I.e. for which $\lambda \in \mathbb{R}$ we have

$$\overrightarrow{(1,1,-1)} \perp \overrightarrow{AP_\lambda}. \qquad P_\lambda = (2,1,1) + \lambda\overrightarrow{(1,1,-1)}$$

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$Q = P_0 = (2,1,1)$

$$0 = \overrightarrow{(1,1,-1)} \cdot \overrightarrow{AP_\lambda} \overset{\downarrow}{=} \overrightarrow{(1,1,-1)} \cdot (\overrightarrow{AQ} + \overrightarrow{QP_\lambda})$$

$$\underset{\uparrow}{=} \overrightarrow{(1,1,-1)} \cdot \overrightarrow{AQ} + \overrightarrow{(1,1,-1)} \cdot \overrightarrow{QP_\lambda}$$

biadditivity

$$= \overrightarrow{(1,1,-1)} \cdot \overrightarrow{(1,1,1)} + \overrightarrow{(1,1,-1)} \cdot (\lambda \overrightarrow{(1,1,-1)})$$

$$\underset{\uparrow}{=} 1 + \lambda(\overrightarrow{(1,1,-1)} \cdot \overrightarrow{(1,1,-1)}) = 1 + \lambda \cdot 3$$

homogeneous on the second component

$$\Leftrightarrow \quad \lambda = -\frac{1}{3}.$$

The point $P_{-\frac{1}{3}} = (2,1,1) - \frac{1}{3}\overrightarrow{(1,1,-1)}$

$= \left(\frac{5}{3}, \frac{2}{3}, \frac{4}{3}\right)$ is nearest to A.

<u>Def 121</u>: (i) The distance (Euclidean distance) between points $P, Q \in X = \mathbb{R}^n$ is defined via

$$d(P,Q) := \|\overrightarrow{PQ}\| = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

(ii) Let $\emptyset \neq Y \subseteq X$ and $A \in X$. The distance between A and Y is defined via

$$\operatorname{dist}(A, Y) := \inf\{d(A, P) \mid P \in Y\}$$

$$= \inf_{P \in Y} d(A, P)$$

(iii) $Y_1, Y_2 \subseteq X$ non-empty. $\operatorname{dist}(Y_1, Y_2) = \inf_{P \in Y_1, Q \in Y_2} d(P, Q)$

Example 122: (i) In Example 120 (b)

we have $d(A, P_\lambda)$

$$= \|\overrightarrow{AP_\lambda}\| = \|\overrightarrow{(1+\lambda, 1+\lambda, 1-\lambda)}\|,$$

$$\left(\text{as } \overrightarrow{AP_\lambda} = \overrightarrow{(1,0,0)(2+\lambda, 1+\lambda, 1-\lambda)} = \overrightarrow{(1+\lambda, 1+\lambda, 1-\lambda)}\right)$$

$$= \sqrt{(1+\lambda)^2 + (1+\lambda)^2 + (1-\lambda)^2}$$

$$= \sqrt{3 + 2\lambda + 3\lambda^2}$$

and $d(A, P_{-\frac{1}{3}}) = \sqrt{3 - \frac{2}{3} + \frac{1}{3}} = 2\sqrt{\frac{2}{3}}$

<u>Check:</u> $\sqrt{\frac{8}{3}} \le \sqrt{3+2\lambda+3\lambda^2}$

$$\left(\begin{array}{l} 3+2\lambda+3\lambda^2 = 3(\lambda^2+2\cdot\frac{1}{3}\lambda+1) \\ = 3\left((\lambda+\frac{1}{3})^2+\frac{8}{9}\right) \ge 3\cdot\frac{8}{9} = \frac{8}{3} \end{array} \right)$$

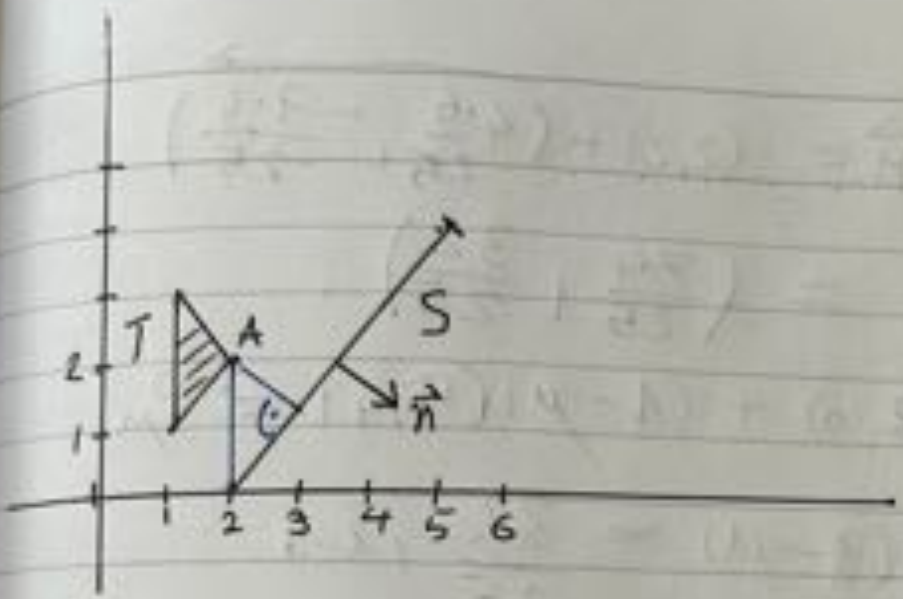
Thus $2\sqrt{\frac{2}{3}} = \mathrm{dist}(A, L)$

(ii) Consider $T, S \subseteq X = \mathbb{R}^2$ defined via

$$T = \{\lambda_1(1,1) + \lambda_2(1,3) + \lambda_3(2,2) \mid 0 \le \lambda_i \le 1,\ i=1,2,3, \text{ and } \lambda_1+\lambda_2+\lambda_3 = 1\}$$

$$S = \{\mu(2,0) + (1-\mu)(5,4) \mid 0 \le \mu \le 1\}$$
$$= \{(5,4) + \mu\overrightarrow{(-3,-4)} \mid 0 \le \mu \le 1\}$$

Compute $\mathrm{dist}(T, S)$.



(Note: The lower edge of T and S are not "parallel.")

<u>Guess:</u> $A := (2,2)$ is closest to S.

The distance of A to the line L through S is

$$\operatorname{dist}(A, L) = \left|\overrightarrow{QA} \cdot \vec{n}\right| \text{ with } Q = (2,0) \text{ and } \vec{n} = \frac{1}{\sqrt{25}}(4,-3)$$

$$= \left|\overrightarrow{(2,0)(2,2)} \cdot \frac{1}{5}(4,-3)\right|$$

$$= \frac{1}{5}\left|\overrightarrow{(0,2)} \cdot \overrightarrow{(4,-3)}\right| = \frac{1}{5}\, 6.$$

The orthogonal projection of A to ~~S~~ L (along $\vec{n}$) is

$$A + \frac{6}{5}\vec{n} = (2,2) + \overrightarrow{\left(\frac{4\cdot 6}{25}, -\frac{3\cdot 6}{25}\right)}$$

$$= \left(\frac{74}{25}, \frac{32}{25}\right)$$

$$= \mu(2,0) + (1-\mu)(5,4) =: P_\mu$$

for $4(1-\mu) = \frac{32}{25}$, i.e.

$$\mu = 1 - \frac{8}{25} = \frac{17}{25} \in [0,1].$$

Thus $P_\mu \in S$.

We need to prove the given.
(Exercise!)

<u>Hint</u>: $\overrightarrow{(2,0)\,(\lambda_1(1,1) + \lambda_2(1,3) + \lambda_3(2,2))}$

$$= \overrightarrow{(\lambda_1(1,1) + \lambda_2(1,3) + \lambda_3(2,2)) - (2,0)}$$

$$\underset{\lambda_1+\lambda_2+\lambda_3=1}{\overset{\uparrow}{=}} \overrightarrow{\lambda_1((1,1)-(2,0)) + \lambda_2((1,3)-(2,0)) + \lambda_3((2,2)-(2,0))}$$

$$= \lambda_1\overrightarrow{(-1,1)} + \lambda_2\overrightarrow{(-1,3)} + \lambda_3\overrightarrow{(0,2)}$$

$\ddagger_{v,v}$

and $\overrightarrow{(-1,1)} \cdot (-\vec{n}) = \overrightarrow{(-1,1)} \cdot \frac{1}{5}\overrightarrow{(-4,3)} = \frac{7}{5} > \frac{6}{5}$

$\overrightarrow{(-1,3)} \cdot (-\vec{n}) = \overrightarrow{(-1,3)} \cdot \frac{1}{5}\overrightarrow{(-4,3)} = \frac{13}{5} > \frac{6}{5}.$

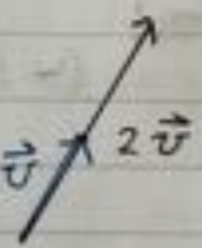
End of Lecture 3rd of Nov 23.

<u>Theorem 123:</u> (Norm properties)

$\|\cdot\| : V \longrightarrow \mathbb{R}^{\geq 0}$ defined via $\|\vec{v}\| := \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$
satisfies the following norm axioms:

(N1) $\forall \vec{v} \in V: \; \|v\| = 0 \Leftrightarrow v = 0.$
("$\|\cdot\|$ is definite")

(N2) $\forall \vec{v} \in V \;\; \forall \lambda \in \mathbb{R}:$

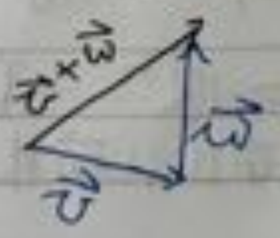


$$\|\lambda \vec{v}\| = |\lambda| \, \|\vec{v}\|$$
("$\|\cdot\|$ is absolute homogeneous")

(N3) $\forall \vec{v}, \vec{w} \in V:$

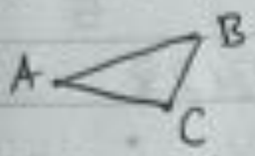
$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.$$

("triangle inequality")



Example 124: $(n=4)$

$A=(0,1,0,1)$, $B=(1,-1,0,1)$, $C=(2,0,1,1)$



$$d(A,B) = \|(1,-2,0,0)\| = \sqrt{5}$$

$$d(B,C) = \|(1,1,1,0)\| = \sqrt{3}$$

$$d(A,C) = \|(2,-1,1,0)\| = \sqrt{6}$$

We have $\sqrt{5}+\sqrt{3} \underset{(*)}{\geq} \sqrt{6}$

Proof:

$$(*) \Leftrightarrow 8 + 2\sqrt{15} \geq 6$$

$$\Leftrightarrow \sqrt{15} \geq -1 \quad \checkmark$$

$\square$

So we have

$$d(A,B) + d(B,C) \geq d(A,C)$$

Lemma 125 (Cauchy-Schwartz inequality)

Let $\vec{v}, \vec{w} \in V = \mathbb{R}^n$. Then

$$|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\,\|\vec{w}\|.$$

Proof: At first we have

$ab \le |ab| \le \frac{1}{2}a^2 + \frac{1}{2}b^2$

for all $a, b \in \mathbb{R}$.

$\Big(0 \le (|a| - |b|)^2 = a^2 + b^2 - 2|a||b|$

$\Rightarrow \quad |ab| \le \frac{1}{2}a^2 + \frac{1}{2}b^2 \Big)$

Take $\vec{v}, \vec{w} \in V$. Then in case of $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$ we get

$$0 = |\vec{v} \cdot \vec{w}| \le 0 = \|\vec{v}\| \, \|\vec{w}\|.$$

and if $\vec{v}, \vec{w} \in V \setminus \{\vec{0}\}$ we get

$$\frac{v_i}{\|\vec{v}\|} \cdot \frac{w_i}{\|\vec{w}\|} \le \frac{1}{2} \frac{v_i^2}{\|\vec{v}\|^2} + \frac{1}{2} \frac{w_i^2}{\|\vec{w}\|^2}.$$

$$\underset{\Sigma}{\Rightarrow} \quad (\vec{v} \cdot \vec{w}) \cdot \frac{1}{\|\vec{v}\|} \cdot \frac{1}{\|\vec{w}\|} \le \frac{1}{2} + \frac{1}{2} \le 1$$

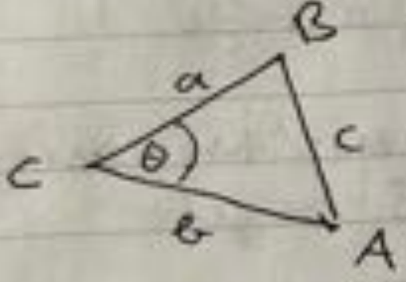
$$\Rightarrow \quad (\vec{v} \cdot \vec{w}) \le \|\vec{v}\| \, \|\vec{w}\|.$$

Same for $-\vec{v}$, so $-(\vec{v} \cdot \vec{w}) \le \|\vec{v}\| \, \|\vec{w}\|$.

$\square$

<u>Remark 126:</u> (geometric meaning of the Cauchy–Schwartz inequality)

We want to define angles using the cosine law.



Put $\vec{v} := \overrightarrow{CB}$ and $\vec{w} := \overrightarrow{CA}$.
We want $\theta \in [0, \pi]$ to satisfy

$$c^2 = a^2 + b^2 - 2ab\cos(\theta).$$

$$\Leftrightarrow \quad \|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\,\|\vec{w}\|\cos(\theta) \quad \Big\} (*)$$

We have $\|\vec{v} + \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})$
$= \vec{v}\cdot\vec{v} - 2\,\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$
$= \|\vec{v}\|^2 - 2\,\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$

Thus $(*) \Leftrightarrow \quad \vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos(\theta)$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}.$$

a, b non-zero

So to define θ we need

$$-1 \leq \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|} \leq 1.$$

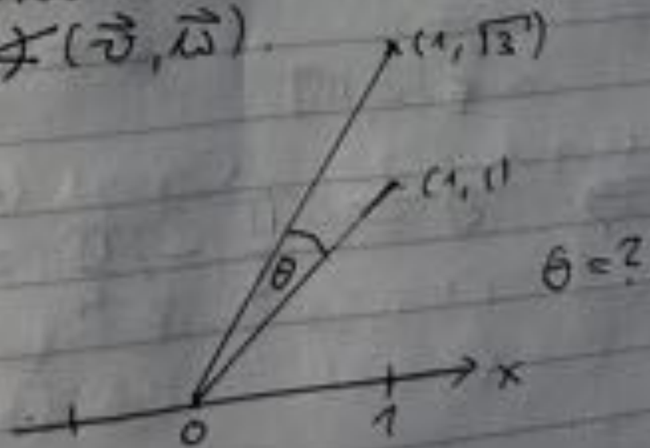
This is provided by the Cauchy-Schwarz inequality.

<u>Def 127:</u> Let $\vec{v}, \vec{w} \in V = \mathbb{R}^n$ non-zero. The number $\theta \in [0, \pi]$ satisfying

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$$ is called

the <u>angle</u> between $\vec{v}$ and $\vec{w}$. We write $\measuredangle(\vec{v}, \vec{w})$.

<u>Example 128:</u>



$$\cos(\theta) = \frac{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}}{\sqrt{2}\cdot\sqrt{4}} = \frac{1+\sqrt{3}}{\sqrt{2}\,2}$$

$\theta = ?$

Another way: $\theta = \beta - \alpha$ with

$$\beta := \sphericalangle\left(\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$
$$\alpha := \sphericalangle\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$\cos\beta = \frac{\begin{pmatrix}1\\\sqrt{3}\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}{\sqrt{4}\,\sqrt{1}} = \frac{1}{2}$$

$$\underset{\beta\in[0,\pi]}{\Longrightarrow} \quad \beta = \frac{\pi}{3}$$

$$\cos\alpha = \frac{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}{\sqrt{2}\,\sqrt{1}} = \frac{1}{\sqrt{2}}$$

$$\underset{\alpha\in[0,\pi]}{\Longrightarrow} \quad \alpha = \frac{\pi}{4}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

Proof of Theorem 12.3:

(definite) $\cdot \|\vec{0}\| = \|(0,\dots,0)\|$
$$= \sqrt{0^2+\dots+0^2} = 0$$

$\cdot$ If $\|\vec{v}\| = 0$ and $\exists i \in \{1,\dots,n\}$: $v_i \neq 0$ then

$$0 = \|\vec{v}\|^2 = v_1^2+\dots+v_n^2 \geq v_i^2 > 0 \;\lightning$$

(absolute homogeneous)
$$\|\lambda\vec{v}\| = \|(\lambda v_1,\dots,\lambda v_n)\|$$
$$= \left(\sum_{i=1}^{n} \lambda^2 v_i^2\right)^{\frac{1}{2}}$$
$$\underset{|\lambda|\geq 0}{\underset{\uparrow}{=}} |\lambda| \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}} = |\lambda|\|\vec{v}\|$$

(Δ-inequality)
$$\|\vec{v}+\vec{w}\|^2 = \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$
$$\underset{\text{Cauchy-Schwartz inequality}}{\leq} \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2$$
$$= (\|\vec{v}\| + \|\vec{w}\|)^2 \quad \square$$

Take $\sqrt{\;}$

Corollary 129: (metric axioms)

The distance map $d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$ satisfies the metric axioms, i.e.

(D1) (being definite)
$d(P,Q) = 0 \iff P = Q$

(D2) (symmetry)
$d(P,Q) = d(Q,P)$

(D3) (Δ-inequality)
$d(P,R) \leq d(P,Q) + d(Q,R)$

Proof: (Exercise) $\square$

Convention 130:
From now on we only work in $V = \mathbb{R}^n$, because we can identify $P \in X = \mathbb{R}^n$ with the vector $\overrightarrow{OP} \in V$.

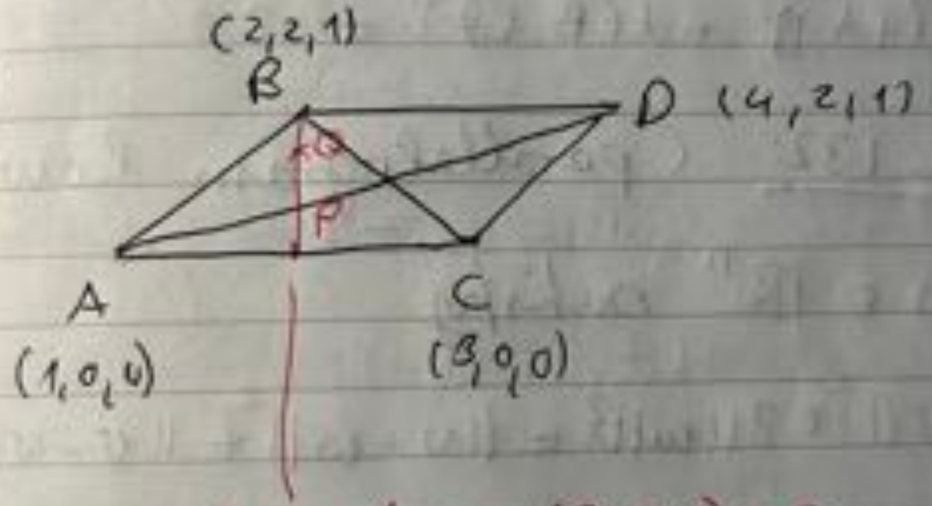
Example 131: Consider the points

$A = (1,0,0)$, $B = (2,2,1)$,
$C = (3,0,0)$, $D = (4,2,1)$

The set $Y := \{ \overbrace{\lambda_1 A + \lambda_2 B + \lambda_3 C + \lambda_4 D}^{\text{"affine combination of points"}} \mid 0 \leq \lambda_i \leq 1 \}$ is a parallelogram

because $\overrightarrow{AC} = \overrightarrow{BD}$ and $\overrightarrow{AB} = \overrightarrow{CD}$

$\mathbb{R}^3 \supseteq$



$\frac{1}{2} A + \frac{1}{2} C = (2,0,0) = P$

$Q = \frac{2}{3} B + \frac{1}{3} P = \frac{2}{3}(2,2,1) + \frac{1}{3}(2,0,0)$

$= \left(\frac{4}{3} + \frac{2}{3}, \frac{4}{3} + 0, \frac{2}{3} + 0\right) = \left(2, \frac{4}{3}, \frac{2}{3}\right)$

Relationship between diagonals and edges of the parallelogram.

length of edges

$d(A,B) = \sqrt{6} = d(C,D)$

$d(B,D) = 2 = d(A,C)$

length of diagonals $d(A,D) = \sqrt{14}$
$d(B,C) = \sqrt{6}$

We have

$$2\, d(A,B)^2 + 2\, d(A,C)^2 = 20$$
$$d(A,D)^2 + d(B,C)^2 = 20$$

Theorem 132: (parallelogram law)

$v, w \in \mathbb{R}^n$ satisfy

$$2\|v\|^2 + 2\|w\|^2 = \|v+w\|^2 + \|v-w\|^2$$

Proof: $\|v+w\|^2 = (v+w)\cdot(v+w)$
$= \|v\|^2 + \|w\|^2 + 2(v \cdot w)$

$\|v-w\|^2 = (v-w)\cdot(v-w)$
$= \|v\|^2 + \|w\|^2 - 2(v \cdot w)$

add $\Rightarrow$ assertion $\square$

Note: The parallelogram law uses the dot product in its proof, but the dot product is not in the statement.

Reason: One can prove that the parallelogram law implies the existence of an Euclidean inner product inducing $\|\cdot\|$ via the following identity.

Prop 133: (Relationship dot product $\leftrightarrow$ inner product)

We have, for $v, w \in \mathbb{R}^n$,

$$v \cdot w = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)$$

Proof exercise. $\square$

Example 134: $v = (1,0,0,2)$
$w = (-1,-2,3,1)$

$v \cdot w = 1$

$\frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right)$
$= \frac{1}{4}\left(\|(0,-2,3,3)\|^2 - \|(2,2,-3,1)\|^2\right)$
$= \frac{1}{4}(22 - 18) = 1.$

Example 135: (ISBN numbers)

ISBN $\hat{=}$ international standard book number

(13 digits)
The last digit is the check digit.

$$d_1 d_2 d_3 - d_4 - d_5 d_6 d_7 d_8 - d_9 d_{10} d_{11} d_{12} - d_{13}$$

Write $d = (d_1, \dots, d_{13})$
The ISBN number has to verify

$$10 \mid d \cdot (1,3,1,3,1,3,\dots,1,3,1)$$

Example: 978-7-5327-7274-?

$$9\cdot1 + 7\cdot3 + 8\cdot1 + 7\cdot3 + 5\cdot1 + 3\cdot3 + 2\cdot1 + 7\cdot3 + 7\cdot1 + 2\cdot3 + 7\cdot1 + 4\cdot3 + d_{check}\cdot 1$$

$$\equiv_{10} (-1)\cdot1 + 1 + (-2) + 1 + 5 + (-1) + 2 + 1 + (-3) + (-4) + (-3) + 2 + d_{check}$$

equal up to a multiple of 10 ("modulo 10")

$$\equiv_{10} -2 + d_{check} \overset{!}{\equiv}_{10} 0$$

$$\Rightarrow d_{check} \equiv_{10} 2, \text{ i.e.}$$

$$d_{check} \in \{2 + 10 \cdot k \mid k \in \mathbb{Z}\}$$

$$\underset{\substack{\uparrow \\ 0 \leq d_{check} \leq 9}}{\Rightarrow} \quad d_{check} = 2$$

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III 4. Orthogonality

Recall: 1) $v \perp w$ if $v \cdot w = 0$,

$v, w \in \mathbb{R}^n$.

2) $v \in \mathbb{R}^n \setminus \{\underline{0}\}$, $w \in \mathbb{R}^n$.

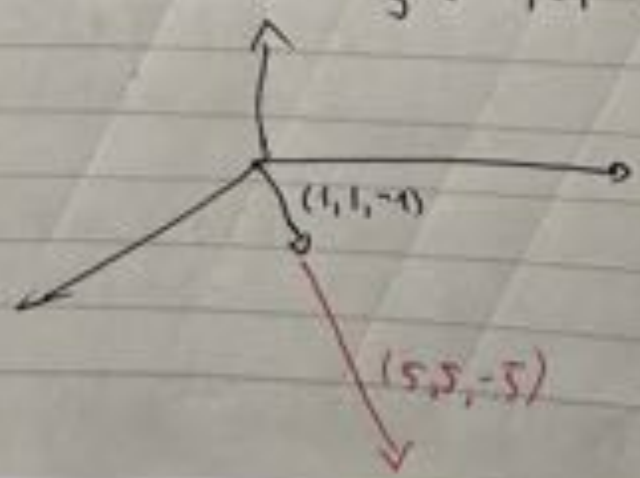
w is called collinear to v if
$\exists\, \lambda \in \mathbb{R}$ $w = \lambda v$.

(we also say w is parallel to v)
Write $w \parallel v$

Example 136:

(a) $(1,1,-1)$ is collinear to $(5,5,-5)$, because

$$(1,1,-1) = \frac{1}{5}(5,5,-5)$$



(b) $\underline{\vec{0}}$ is collinear to every vector.

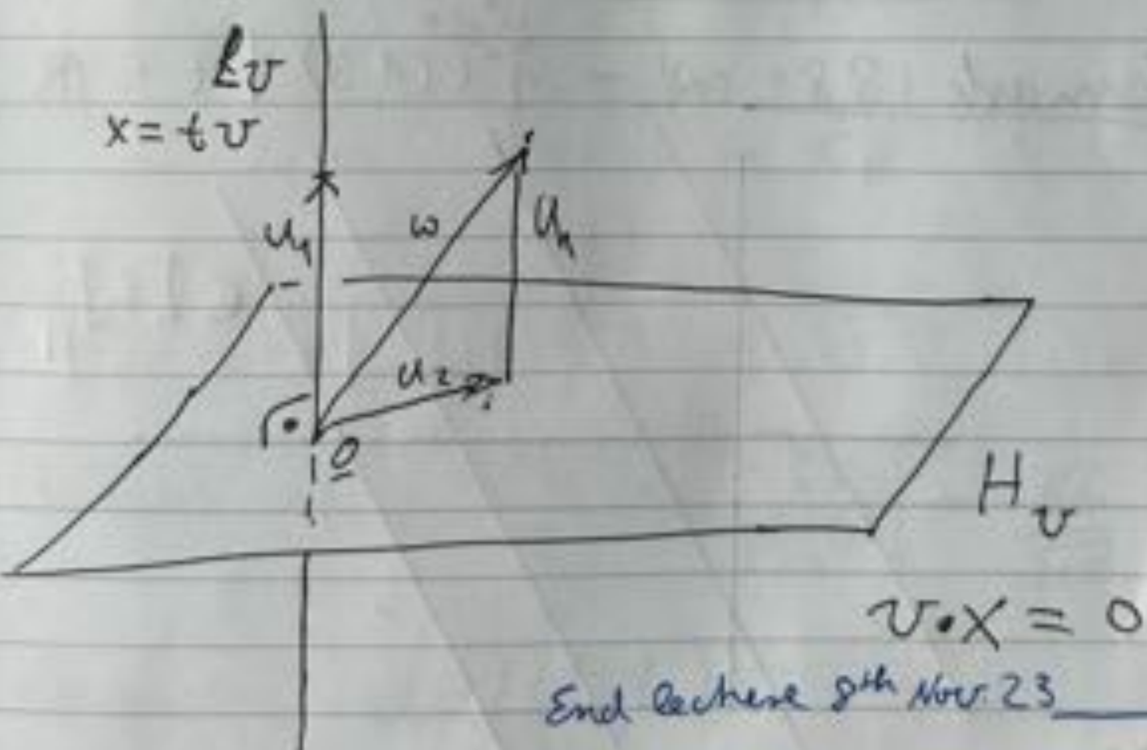
(c) The "<u>Projection theorem</u>"

(Proposition 117) states that for $v,$

$\in \mathbb{R}^n \setminus \{\vec{0}\}$ and $w \in \mathbb{R}^n$

$\exists!\, u_1, u_2 \in \mathbb{R}^n : w = u_1 + u_2$

$\wedge\ u_1 \parallel v \ \wedge\ u_2 \perp v.$



End lecture 8th Nov. 23

$\rightsquigarrow$ See example class.

Notation: $u_1 = \mathrm{proj}_v(w) = (w \cdot v_{norm})\, v_{norm}$

H_1, H_2

Def 137: Two hyperplanes are called parallel ($H_1 \parallel H_2$) if

$\exists\, v \in \mathbb{R}^n \setminus \{\underline{0}\}$: v is normal to

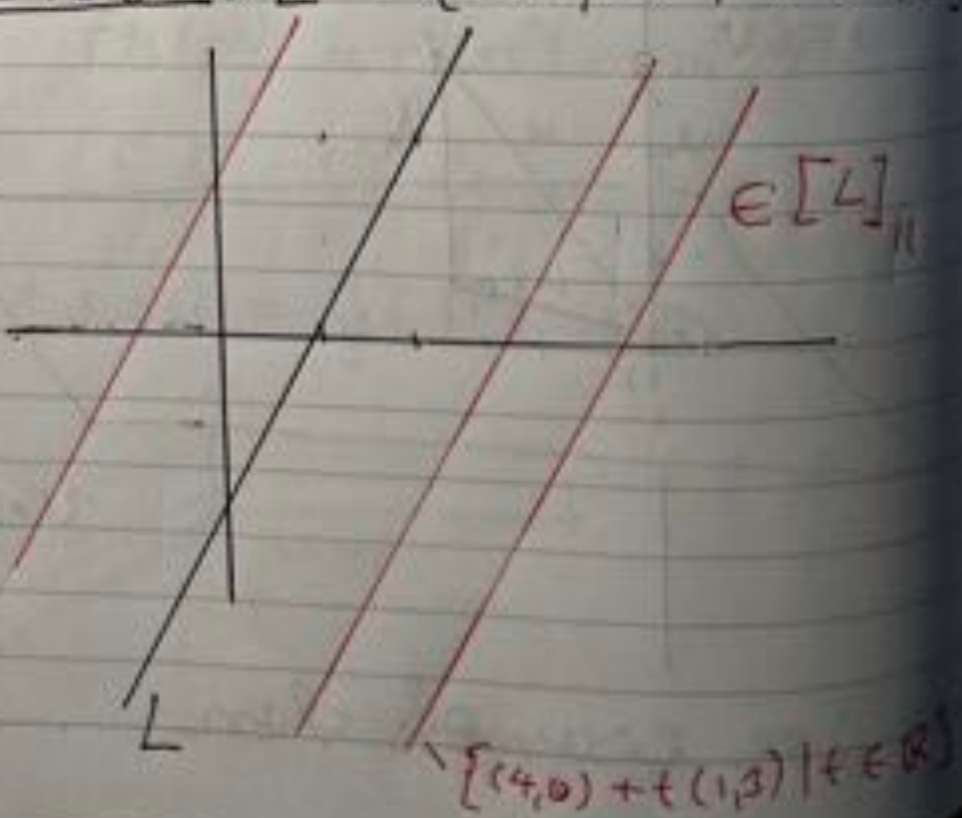
H_1 and H_2.

The set $[H_1]_{\parallel} = \{ H \subseteq \mathbb{R}^n \mid$

H hyperplane $\parallel H_1 \}$ is called

the parallel class of H_1.

Remark 138: (a) $L = \{ (1,0) + t(1,3) \mid t \in \mathbb{R} \}$



(b) Give an example for a parallel class of planes in $\mathbb{R}^3$.

(c) In (a): $P_0 = (5,-1) \in \underbrace{L + (4,-1)}_{=:L_0}$

Consider normal eqn:

$L: \ -3x + y = -3$
$L_0: \ -3x + y = -16$

normalize $\overrightarrow{(-3,1)}$: $\frac{1}{\sqrt{10}}(-3,1)$

$L: \ -\frac{3}{\sqrt{10}}x + \frac{1}{\sqrt{10}}y = \frac{-3}{\sqrt{10}}$

$L_0: \ \frac{-3}{\sqrt{10}}x + \frac{1}{\sqrt{10}}y = -\frac{16}{\sqrt{10}}$

What is $\left| \frac{-16}{\sqrt{10}} - \frac{(-3)}{\sqrt{10}} \right| = \frac{13}{\sqrt{10}}$?

Answer: $\operatorname{dist}(P_0, L) = \frac{13}{\sqrt{10}}$.

<u>Pf:</u> $\mathrm{dist}(P_0, L) = \left| \frac{1}{\sqrt{10}} (-3,1) \cdot \overrightarrow{(1,0)P_0} \right|$

$$= \frac{1}{\sqrt{10}} \left| (-3,1) \cdot (4,-1) \right|$$

$$= \frac{1}{\sqrt{10}} \left| (-12 - 1) \right| = \frac{13}{\sqrt{10}}. \quad \square$$

<u>Theorem 139</u> (Distance theorem)

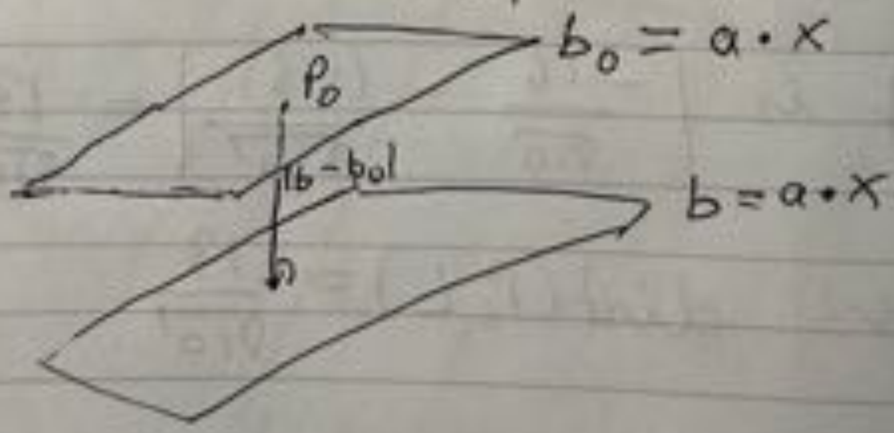
Let H be a hyperplane in $\mathbb{R}^n$ given by

$$a \cdot x = b \quad , \; a \in \mathbb{R}^n \setminus \{0\}, \; b \in \mathbb{R}$$

such that $\|a\| = 1$.
Take $P_0 \in \mathbb{R}^n$ with coordinates

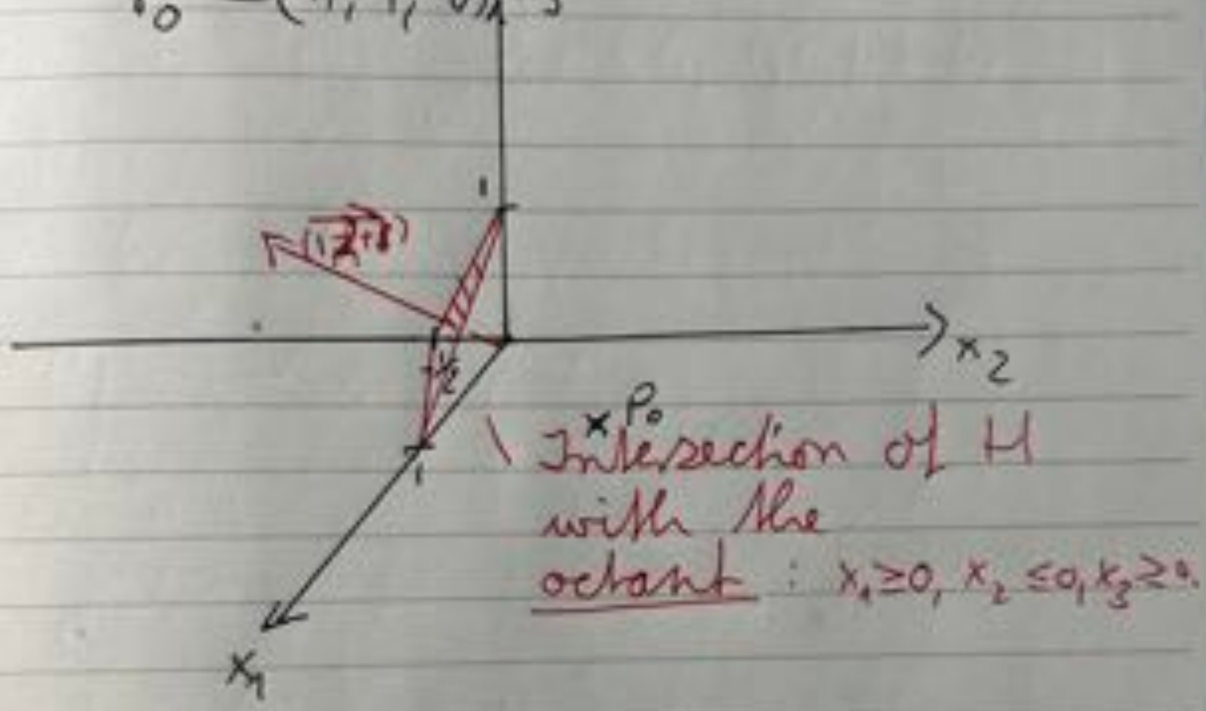
$$x_0 = (x_{01}, x_{02}, \ldots, x_{0n}).$$

Then $\mathrm{dist}(P_0, H) = |b - a \cdot x_0|$



Example 140: $\mathbb{R}^3 \supseteq H: (1,-2,1)\bullet x = 1$

$P_0 = (1,1,0)$



$$\operatorname{dist}(P_0, H) = \left| \frac{1}{\sqrt{6}} - (1,1,0)\bullet\frac{1}{\sqrt{6}}(1,-2,1) \right|$$

$$= \frac{1}{\sqrt{6}}\left|1-(-1)\right| = \frac{2}{\sqrt{6}}.$$

III 5. Geometry of linear systems

Def 141: A *line* in $\mathbb{R}^n$ is a set of the form $L(P_0,P_1) := \{ tP_0 + (1-t)P_1 \mid t \in \mathbb{R} \}$ with $P_0, P_1 \in \mathbb{R}$. ($P_0 \neq P_1$)

A *plane* in $\mathbb{R}^n$ is a set of the form

$$E(P_0,P_1,P_2) := \{ tP_0 + sP_1 + (1-t-s)P_2 \mid t, s \in \mathbb{R} \}$$

for $P_0, P_1, P_2 \in \mathbb{R}^n$ ~~s.t.~~ pairwise different such that

$$\overrightarrow{P_0P_1} \not\parallel \overrightarrow{P_0P_2}$$

Example 142:

(a) $P_0 = (1,0,0)$, $P_1 = (1,1,1)$ in $\mathbb{R}^3$

$L(P_0,P_1)$

(b) $Q_0 = (1,0,0,0)$, $Q_1 = (1,1,0,0)$
$Q_2 = (0,0,1,1)$ in $\mathbb{R}^4$.

$E(Q_0,Q_1,Q_2)$

Remark 143: (ways to describe lines and planes.)

We refer to Example 142 (a), (b).

way	line	plane
vector form	$P_0 + t\vec{v},\ t\in\mathbb{R}$ (with $\vec{v}\neq 0$) Ex: $(1,-1,-1)+t\overrightarrow{(0,1,1)}$ $t\in\mathbb{R}$	$P_0 + tw + su$, $t,s\in\mathbb{R}$ (with $w,u\neq 0$ and $w \nparallel u$) Ex: $(1,0,0,0)$ $+t\overrightarrow{(0,1,0,0)}+s\overrightarrow{(-1,0,1,1)}$ $t,s\in\mathbb{R}$
using parametric equations	$x_1 = x_{01} + tv_1$ $x_2 = x_{02} + tv_2$ $\vdots$ $x_n = x_{0n} + tv_n$ Ex: $x_1 = 1,\ x_2 = -1+t$ $x_3 = -1+t$	$x_1 = x_{01} + tw_1 + su_1$ $\vdots$ $x_n = x_{0n} + tw_n + su_n$. Ex: $x_1 = 1+(-1)s,\ x_2 = t$ $x_3 = s,\ x_4 = s$

	$L = H_1 \cap \dots \cap H_{n-1}$	$E = H_1 \cap \dots \cap H_{n-2}$
as intersection of hyper-planes	$\cancel{L = H_1 \cap \dots \cap H_{n-1}}$	$\cancel{E = H_1 \cap \dots \cap H_{n-2}}$
	Ex:	
	$L = H_1 \cap H_2$	$E = H_1 \cap H_2$
	$H_1 = \{x \in \mathbb{R}^3 \mid (1,0,0) \cdot x = 1\}$	$H_1 = \{x \in \mathbb{R}^4 \mid (1,0,1,0) \cdot x = 1\}$
	$H_2 = \{x \in \mathbb{R}^3 \mid (0,1,-1) \cdot x = 0\}$	$H_2 = \{x \in \mathbb{R}^4 \mid (0,0,1,-1) \cdot x = 0\}$

Example 144: (a) $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$

$$\mathrm{sol}(A,b) := \{x \in \mathbb{R}^n \mid Ax = b\}$$

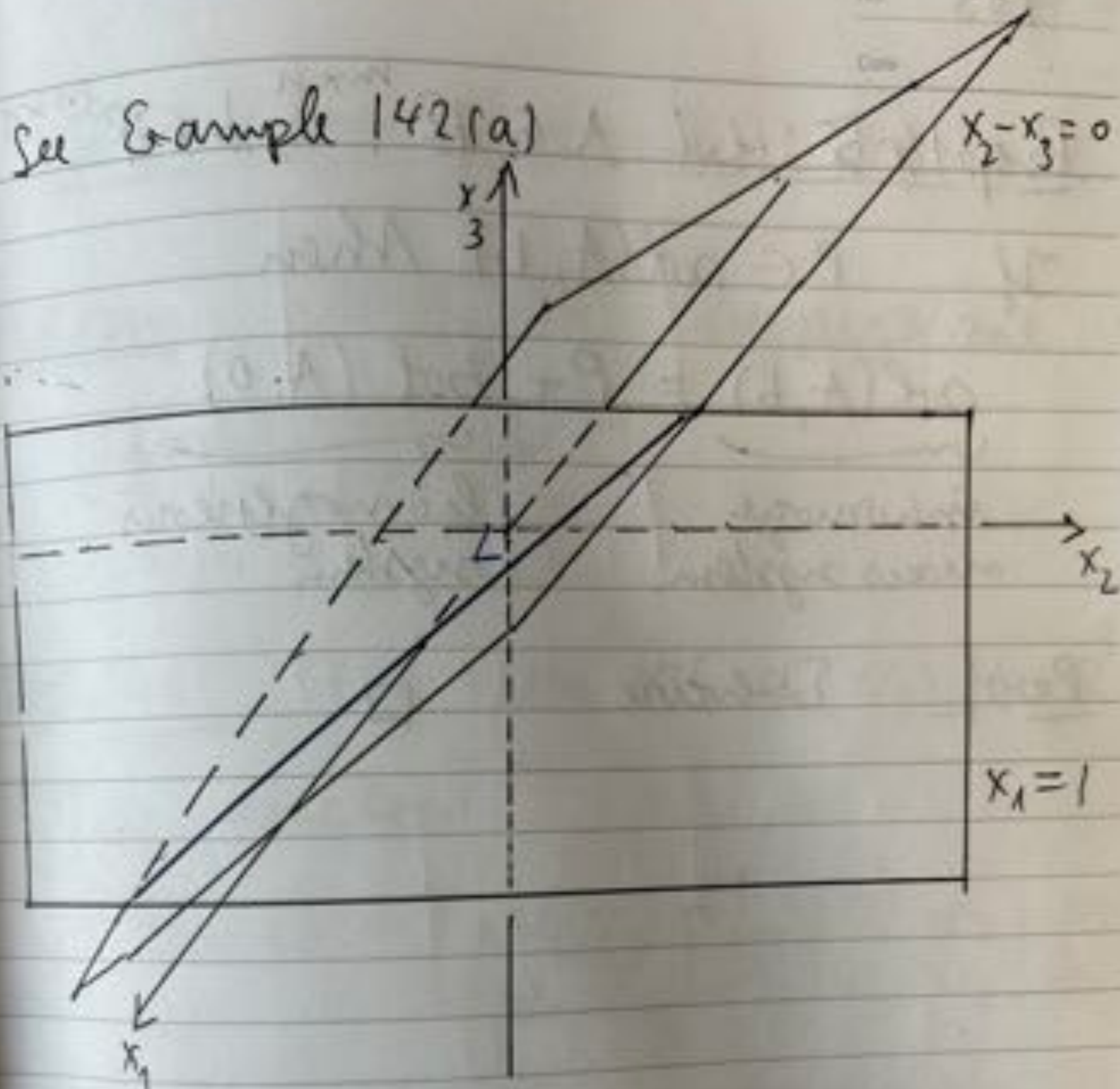
$$= \mathrm{sol}(r_1, b_1) \cap \mathrm{sol}(r_2, b_2) \cap \dots \cap \mathrm{sol}(r_m, b_m)$$

Where $A = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}$ rows.

Suppose that all $r_i \neq (0, \dots, 0)$

$\Rightarrow$ all $\mathrm{sol}(r_i, b_i)$ are hyperplanes

See Example 142(a)

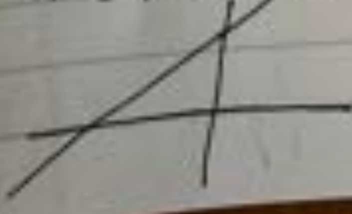


$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$sol(A, b) = sol((1,0,0), 1) \cap sol((0,1,-1), 0)$$

(ii)

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad sol(A, b) = \emptyset$$



Prop 145: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$

If $P \in sol(A, b)$ then

$$\underbrace{sol(A,b)}_{\text{inhomogeneous system.}} = P + \underbrace{sol(A,0)}_{\text{homogeneous system}}$$

Proof: Exercise $\square$

III 6. Cross product

Problem: $H \subseteq \mathbb{R}^3$ hypyplane

Find $\vec{n} \perp H$
$\neq 0$

Def 146: Let $v = (v_1, v_2, v_3), w \in \mathbb{R}^3$

The vector

$$v \times w := \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$$

is called the cross product of v and w.

⌈ If we have $\vec{i} := (1,0,0)$ $\vec{j} := (0,1,0)$
$\vec{k} = (0,0,1)$. Then formally

$$v \times w = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 147:

(a) $\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$

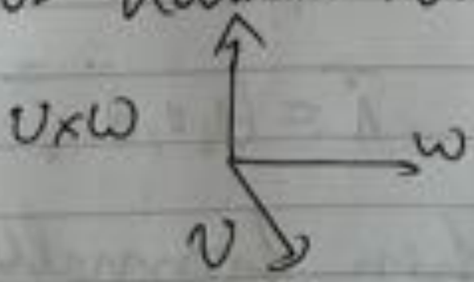
$= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$= \vec{k} = (0,0,1)$

$\vec{j} \times \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\vec{k}$

$= (0,0,-1)$

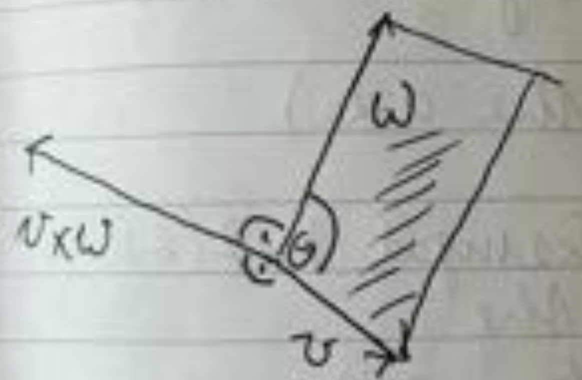
Right hand rule



(b) $v = (1, 1, 0)$, $w = (0, -2, 2)$

$$v \times w = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & -2 & 2 \end{vmatrix}$$

$$= (2, -2, 2)$$



$$\text{area} = \|v\|\,\|w\| \sin\theta$$
$$= \sqrt{2} \cdot 2\sqrt{2} \cdot \frac{\sqrt{3}}{2}$$
$$= \underline{\underline{2\sqrt{3}}}.$$

(where $\sin\theta = \sqrt{1 - \cos^2\theta}$

$$= \sqrt{1 - \left(\frac{v \cdot w}{\|v\|\,\|w\|}\right)^2}$$

$$= \sqrt{1 - \left(\frac{-2}{\sqrt{2}\, 2\sqrt{2}}\right)^2}$$

$$= \frac{\sqrt{3}}{2} \quad)$$

$$\|v \times w\| = \sqrt{4+4+4} = 2\sqrt{3}.$$

Prop 148: Let $v, w \in \mathbb{R}^3 \setminus \{\underline{0}\}$

Then $v \times w$ has direction given by the right hand rule and

$\|v \times w\|$ = area of the parallelogramme spanned by v and w.

Proof: (only for the area)

By the idea of Example 147(b) it follows from the Lagrange identity, see next proposition $\square$

Prop 149: (crossproduct and dot product.)

$u, v, w \in \mathbb{R}^3$

(a) u and v are $\perp$ to $u \times v$

(b) $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

(c) (Lagrange identity)

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$$

<u>Proof:</u> (a) $u \cdot (u \times v) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0$

$v \perp (u \times v)$ similarly

(b) Exercise.

(c) $\|u \times v\|^2 = (u_2 v_3 - v_2 u_3)^2 + (u_1 v_3 - v_1 u_3)^2$

$+ (u_1 v_2 - u_2 v_1)^2$

$= u_2^2 v_3^2 + v_2^2 u_3^2 - 2 u_2 u_3 v_2 v_3$

$+ u_1^2 v_3^2 + u_3^2 v_1^2 - 2 u_1 u_3 v_1 v_3$

$+ u_1^2 v_2^2 + u_2^2 v_1^2 - 2 u_2 u_3 v_2 v_3$

$= \|u\|^2 \|v\|^2 - u_1^2 v_1^2 - u_2^2 v_2^2 - u_3^2 v_3^2 - \sum_{i \neq j} (u_i v_i)(u_j v_j)$

$= \|u\|^2 \|v\|^2 - (u \cdot v)^2$. $\square$

Prop 150. Given $u, v, w \in \mathbb{R}^3$

The map $[\ ,\ ,\ ] : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

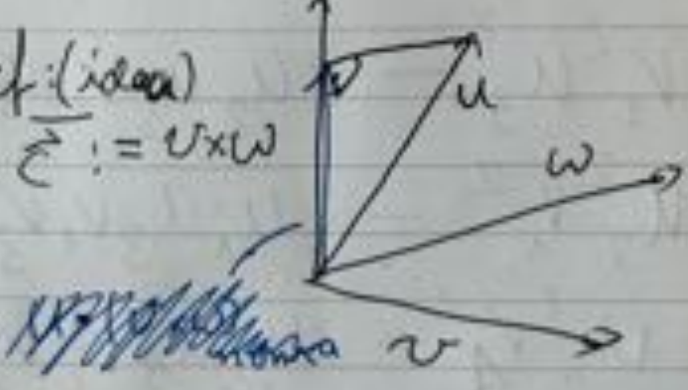
$$(u,v,w) \longmapsto \underset{\substack{\shortparallel \\ [u,v,w]}}{u \cdot (v \times w)} = \begin{vmatrix} u \\ v \\ w \end{vmatrix}$$

is called <u>scalar triple product</u>

and $[u,v,w]$ is the signed volume of the parallelepiped spanned by u, v, w.

Proof: (idea)

$\vec{c} := v \times w$



$(u \cdot \vec{c}_{norm})\, \vec{c}_{norm} =: \vec{d} = proj_c(u)$

So $[u,v,w] = \|\vec{d}\| \cdot \|\vec{c}\| =$ volume of the parallelepiped $\square$