LINEAR ALGEBRA 1
PROBLEM SHEET 5

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Problem 1 (20 points, vector space axioms)**.** Prove Proposition 106, except of (ass +) and (ldist).

Problem 2 (10+10 points, hyperplanes)**.** Consider the following two pairs of hyperplanes in $X = \mathbb{R}^3$. If they intersect then compute the angle between them. If they don't intersect then compute their distance.

(i) $\mathrm{H}_1 := \{(x_1, x_2, x_3) \in \mathrm{X} \mid\ x_1 + 2x_2 + x_3 = 1\}$ and
$\mathrm{H}_2 := \{t_{\vec{v}}(\mathrm{P}) \mid\ \vec{v} \in \mathbb{R}\overrightarrow{(1, 1, -3)} + \mathbb{R}\overrightarrow{(1, -1, 1)}\},\ \mathrm{P} = (0, 1, 0)$

(ii) H_1 and $\mathrm{H}_3 := \{t_{\vec{v}}(\mathrm{P}) \mid\ \vec{v} \in \mathbb{R}\overrightarrow{(1, 1, 0)} + \mathbb{R}\overrightarrow{(1, 0, 1)}\}$

Remark: The angle θ between non-zero vectors $\vec{v}, \vec{w}$ is given by

$$\cos(\theta) = \frac{\vec{v} \bullet \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$$

Problem 3 (15 points, polytope)**.** Consider the n-space with point space $\mathrm{X} = \mathbb{R}^n$ and vector space $\mathrm{V} = \mathbb{R}^n$.

(i) Let P be a point of X and L be an affine line contained in X which does not contain P. Let Q be a point on L. Prove that for every $\delta > 0$ there is a point $\mathrm{Q}_\delta \neq \mathrm{Q}$ on L such that $d(\mathrm{P}, \mathrm{Q}_\delta) > d(\mathrm{P}, \mathrm{Q})$ and $d(\mathrm{Q}, \mathrm{Q}_\delta) < \delta$.

(ii) $(n = 3)$ Consider the following subset:

$$\mathrm{Y} := \{(1, 1, 1) + \lambda_1\overrightarrow{(0, 2, 1)} + \lambda_2\overrightarrow{(-1, 2, 1)} \mid\ \lambda_1, \lambda_2 \in [0, 1]\}$$

Find in Y all pairs of points $\mathrm{P}_0, \mathrm{Q}_0$ of maximal distance to each other, i.e. such that

$$d(\mathrm{P}_0, \mathrm{Q}_0) = \sup\{d(\mathrm{P}, \mathrm{Q}) \mid\ \mathrm{P}, \mathrm{Q} \in \mathrm{Y}\}$$

. (You have to prove your result. Hint: (i) could help.)

Problem 4 (20 points, distance between two lines)**.** Consider the lines:

$$\mathrm{L}_1 := (1, 1, 1, 1) + \mathbb{R}\overrightarrow{(2, -1, 3, 1)},\ \mathrm{L}_2 := (1, 0, 0, 0) + \mathbb{R}\overrightarrow{(1, -1, 1, 1)}$$

Find the distance between those lines and find all pairs of points $\mathrm{A} \in \mathrm{L}_1, \mathrm{B} \in \mathrm{L}_2$ which are closest to each other, i.e. such that $d(\mathrm{A}, \mathrm{B}) = \mathrm{dist}(\mathrm{L}_1, \mathrm{L}_2)$..

Date: Please hand in before the lecture by **8th of November 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.