II.3 Applications

Another application of the concept of determinant is Cramer's rule

Theorem 93 (Cramer's rule)

Suppose $A \in \mathbb{R}^{m\times m}$ is invertible and $\underline{b} \in \mathbb{R}^{m\times 1}$.
Then the unique solⁿ of $Ax=b$ is given by

$$\cancel{x_i = \det(\mathrm{col}_1(A), \mathrm{col}_2(A), \ldots, \mathrm{col}_{i-1}(A), b, \ldots)}$$

$$x_1 = \frac{\det(A_1)}{\det(A)}, \ldots, \quad x_m = \frac{\det(A_m)}{\det(A)}$$

where

$$A_i = \left(\mathrm{col}_1(A), \ldots, \mathrm{col}_{i-1}(A), \underline{b}, \mathrm{col}_{i+1}(A), \ldots, \mathrm{col}_m(A) \right.$$

i.e. A_i is obtained by replacing the ith column of A by $\underline{b}$.

Proof: We have

$$\det(A_i) = \sum_{j=1}^{m} b_j \, C_{ji}(A)$$

$$\Rightarrow \begin{pmatrix} \det(A_1) \\ \vdots \\ \det(A_m) \end{pmatrix} = \mathrm{Adj}(A)\, \underline{b}$$

$$\Rightarrow Ax = \frac{1}{\det(A)} A \,\mathrm{Adj}(A)\, \underline{b}$$

$$= \frac{1}{\det(A)} \det(A)\, I_m \underline{b}$$

$$= \underline{b}. \qquad \square$$

Remark 94: This way has high complexity if m is big.

Example 95:

$$\left(\underbrace{\begin{matrix} 2 & 1 & 1 \\ 1 & -1 & 3 \\ 5 & 7 & 1 \end{matrix}}_{A} \;\middle|\; \underset{b}{\begin{matrix} 1 \\ 0 \\ 1 \end{matrix}} \right)$$

$$\det(A) = 20 - (38) = -18$$

$$\det(A_1) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 1 & 7 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 6 & 0 \end{vmatrix} = -18$$

$$\det(A_2) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 5 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 3 & 1 & 0 \end{vmatrix} = 9$$

$$\det(A_3) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 5 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 3 & 6 & 1 \end{vmatrix} = 9$$

So the sol^n is $\begin{pmatrix} -18 \\ 9 \\ 9 \end{pmatrix} \cdot \dfrac{1}{-18} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

Sign of a permutation:

Def 96: $P \in \mathbb{R}^{n \times n}$ is called permutation matrix if

(i) $\forall$ row $\exists!$ non-zero entry in the row,

(ii) $\forall$ column $\exists!$ — " — column.

and (iii) all non-zero entries are 1.

Example 97:

(a) $$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = P$$

P is a product of E_{jk} s

(They are also called transposition matrices)

$$P \xrightarrow{\binom{II}{I}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{\binom{IV}{II}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & 1 & \end{pmatrix}$$

$$\xrightarrow{\binom{IV}{III}} I_4$$

$$\Rightarrow \quad E_{34} E_{24} E_{12} P = I_3$$

$$\Rightarrow \quad P = E_{12} E_{24} E_{34}$$

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(b) $P = I_m$

<u>Prop 98:</u> A permutation matrix can be writen as a product of transposition matrices

$$P = T_1 \circ \dots \circ T_\ell .$$

Then: 1) ℓ is even if $\det(P) = 1$
2) ℓ is odd if $\det(P) = -1$.

<u>Proof:</u>

$$\det(P) = \det(T_1) \cdots \det(T_\ell) = (-1)^\ell$$

$\square$

Therefore it <u>cannot</u> happen that P can be decomposed into odd many and even many transpositions.

P permutes the entries of a vector.

<u>Ex: 99:</u> $P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & & 0 \end{pmatrix}$ —

$$P\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}x_4\\x_1\\x_3\\x_2\end{pmatrix}$$

So we get a map $\sigma_P : \{1,2,3,4\} \to \{1,2,3,4\}$ such that

$\sigma_P(1) = 2$, $\sigma_P(2) = 4$, $\sigma_P(3) = 3$
$\sigma_P(4) = 1$

non-zero entries of P are at
$(\sigma_P(i), i)$, $i = 1,2,3,4$.., i.e.

$(2,1), (4,2), (3,3), (1,4)$.

Conversely a bijection $\sigma : \{1,..,n\} \to \{1,..,n\}$ defines a permutation matrix P_σ with 1's at $(\sigma(1),1), \dots, (\sigma(n),n)$.

Def 100: A bijection $\sigma : \{1,..,n\} \to \{1,..,n\}$ is called a permutation.
The sign of σ is defined via

$$\operatorname{sgn}(\sigma) := \det(P_\sigma).$$

example: $\sigma : \{1,\dots,4\} \to \{1,\dots,4\}$.

$\sigma(1) = 2$

$\sigma(2) = 3$

$\sigma(3) = 4$

$\sigma(4) = 1$

$$P_\sigma = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix}$$

$$\operatorname{sgn}(\sigma) = \det(P_\sigma) = -\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} = -1.$$