Appendix of Chapter I

We prove

Thm 25: Let $A \in \mathbb{R}^{m\times n}$ and R, R' be reduced row echelon forms of A. Then $R = R'$.

Proof: Let $j_1, \dots, j_\ell$ and $j_1', \dots, j_{\ell'}'$ be the "pivot" columns of R and R', respectively. We show three claims

Claim 1: $\ell = \ell'$. Claim 2: $(j_1, \dots, j_\ell) = (j_1', \dots, j_{\ell'}')$

Claim 3: $R = R'$

Proof of Claim 1: Assume $\ell > \ell'$.

R^* is a reduced row echelon form of R'. Thus $\exists$ E a product of elementary matrices such that $ER' = R$.

We reduce the equation:

Consider $$P = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathbb{R}^{n\times \ell}$$

where the 1's are in the rows $j_1, j_2, \dots, j_\ell$.

A2

We obtain $\quad RP = \begin{pmatrix} I_\ell \\ 0 \end{pmatrix} \begin{matrix} \in \mathbb{R}^{m \times \ell} \\ \}m-\ell \end{matrix}$

and $\quad R'P = \begin{pmatrix} * \\ 0 \cdots 0 \\ 0 \end{pmatrix} \begin{matrix} \in \mathbb{R}^{m \times \ell} \\ \}m-\ell+1 \end{matrix}$

because $\ell' > \ell$. Further we use

$D = \begin{pmatrix} I_\ell & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \in \mathbb{R}^{m \times m}$ and we

obtain

$$\begin{pmatrix} I_\ell \\ 0 \end{pmatrix} = D \begin{pmatrix} I_\ell \\ 0 \end{pmatrix} = DER'P$$

$$\underset{\underset{R'P = DR'P}{\uparrow}}{=} (DED)\, R'P$$

$$= \ell\{ \underbrace{\begin{pmatrix} \tilde{\hat{E}} & 0 \\ 0 & 0 \end{pmatrix}}_{\ell} \begin{pmatrix} * \\ 0 \cdots 0 \\ 0 \end{pmatrix} \} =: \tilde{R}'$$

Thus $\tilde{E}\tilde{R}' = I_\ell$

Theorem 44 (which does not use Thm 25) $\Rightarrow$ $\tilde{R}'\tilde{E} = I_\ell$

$\lightning$, because $\tilde{R}'E$ has a zero row.

<u>Proof of Claim 2:</u> By induction on n using Claim 1.

(BC) <u>base case</u>: $(n=1)$. Claim 1 $\Rightarrow \ell = \ell'$. Thus
$$R = R' = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{m\times 1} \text{ or } R = R' = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{m\times 1}.$$

(IS) <u>induction step</u> $(n \longrightarrow n+1)$: Write $R = (R_1|c)$ and $R' = (R_1'|c')$ with $c, c' \in \mathbb{R}^{m\times 1}$ and $R_1, R_1' \in \mathbb{R}^{m\times n}$.

- If $j_\ell, j'_{\ell'} < n+1$ then $(j_1, \dots, j_\ell) = (j_1', \dots, j_\ell')$ by induction hypothesis (IH) on R_1, R_1'.
- Similarly for the case $j_\ell = j'_\ell = n+1$.
- Assume $j'_\ell = n+1$ and $j_1 \leq n$.

A4

~~The~~ The number of non-zero rows in R is smaller than the number of those in R'. $\lightning$ to Claim 1.

$\square$ (Claim2)

Proof of Claim 3: We use the matrices of the proof of Claim 1.

We have $\tilde{E}\tilde{R}' = I_\ell$ and

$\tilde{R}' = I_\ell$ by Claim 1 and Claim 2.

thus $\tilde{E} = I_\ell$ and

$$\begin{pmatrix} \tilde{E} & 0 \\ 0 & 0 \end{pmatrix} R' = R$$

$$\Rightarrow R' = R$$

$\square$