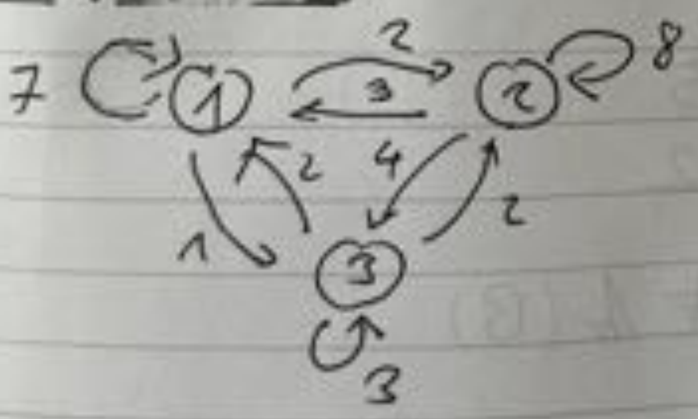
Chapter II Determinant and trace of a square matrix.

II.1. Trace

Definition 60: Let $A \in \mathbb{R}^{m \times m}$. The number $tr(A) := \sum_{i=1}^{m} a_{ii}$ is called the trace of A.

Example 61: (a) See Example 52:



$$A = \begin{pmatrix} 7 & 2 & 1 \\ 3 & 8 & 4 \\ 2 & 2 & 3 \end{pmatrix} \qquad tr(A) = 7 + 8 + 3 = 18$$

So 18000 people do not go to another city for work.

(b) Is there a matrix $C \in \mathbb{R}^{m \times m}$ $(m=3)$ invertible such that

$$\underbrace{\begin{pmatrix} 1 & 2 & 7 \\ -1 & 3 & 1 \\ 5 & 2 & 1 \end{pmatrix}}_{A} = C \underbrace{\begin{pmatrix} 1 & 5 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix}}_{B} C^{-1},$$

i.e. can we get A from B by coordinate transformation

$$y = Cx \quad , y, x \in \mathbb{R}^{m \times 1} \ ?$$

$$tr(A) = 5$$
$$tr(B) = 2$$

$$\Rightarrow tr(A) \neq tr(B).$$

<u>Prop. 62:</u> $\forall A, C \in \mathbb{R}^{m \times m}$:

$$tr(AC) = tr(CA).$$

<u>Proof:</u> $\operatorname{tr}(AC) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} c_{ji}$

$$= \sum_{j=1}^{m} \sum_{i=1}^{m} c_{ji} a_{ij} = \operatorname{tr}(CA) \quad \square$$

$+, \cdot$
commutative
in $\mathbb{R}$

<u>How</u> does this answer Example 61(b)?

$$\operatorname{tr}(CBC^{-1}) \underset{\text{Prop. 62}}{=} \operatorname{tr}(BC^{-1}C) = \operatorname{tr}(BI_3)$$

$$= \operatorname{tr}(B) \neq \operatorname{tr}(A)$$

for all $C \in \mathbb{R}^{3\times 3}$ invertible.

<u>Answer:</u> No.

<u>Example 63:</u>

(a) $\operatorname{tr}\left(\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 5 \end{pmatrix}\right)$

$\operatorname{tr}\begin{pmatrix} 3 & 2 \\ 1 & 7 \end{pmatrix} = 3 + 7 = 10$

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$$tr\left(\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\right) + tr\left(\begin{pmatrix}2 & 1\\ 0 & 5\end{pmatrix}\right)$$

$$= 3 + 7$$

So $tr\left(\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix} + \begin{pmatrix}2 & 1\\ 0 & 5\end{pmatrix}\right)$

$$\parallel$$

$$tr\left(\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\right) + tr\left(\begin{pmatrix}2 & 1\\ 0 & 5\end{pmatrix}\right).$$

Is this a coincident?

(c)

$$tr\left(\lambda\begin{pmatrix}1 & 3 & 2\\ 2 & 1 & -1\\ -10 & 5 & +3\end{pmatrix}\right) = \lambda + \lambda + \lambda 3$$

$$= 5\lambda$$

$$\lambda\, tr\begin{pmatrix}1 & 3 & 2\\ 2 & 1 & -1\\ -10 & 5 & 3\end{pmatrix} = \lambda \cdot 5.$$

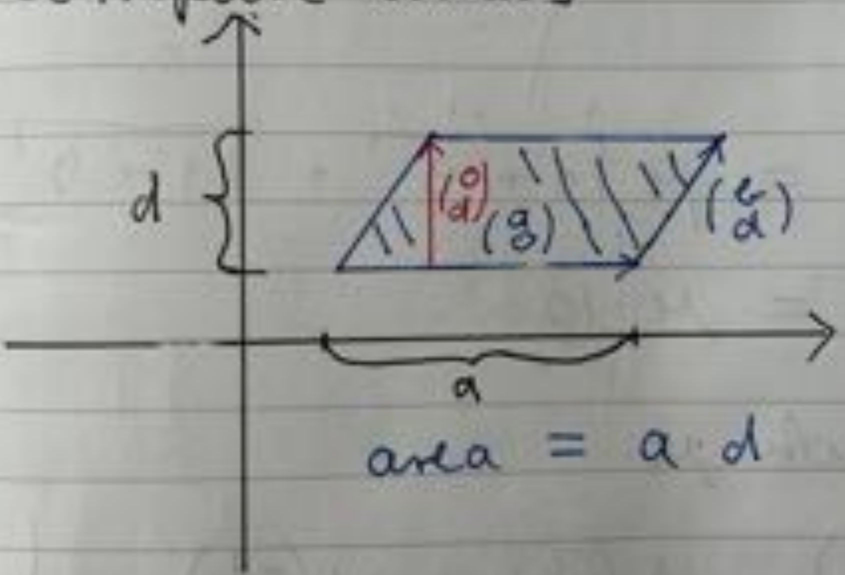
II 2. Determinant of a matrix

Another invariant of a matrix is the determinant.

II 2.1. Motivation (volume of a parallelotope)

We compute areas

(1)



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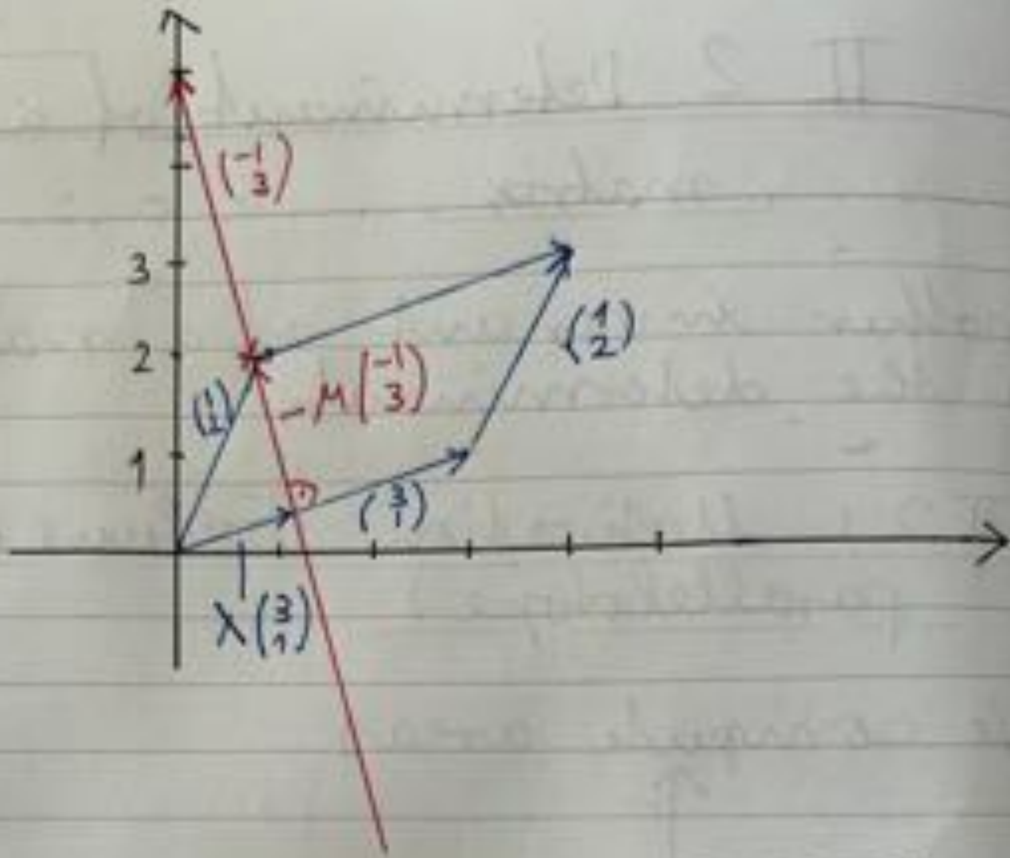


$$\text{area} = \mu \sqrt{1+9} \cdot \sqrt{1+9}$$

$$= \mu \cdot 10$$

Compute μ:

$$\begin{pmatrix}1\\2\end{pmatrix} = \lambda\begin{pmatrix}3\\1\end{pmatrix} + \mu\begin{pmatrix}-1\\3\end{pmatrix} \quad \Big| \, (-1 \;\; 3)\circ$$

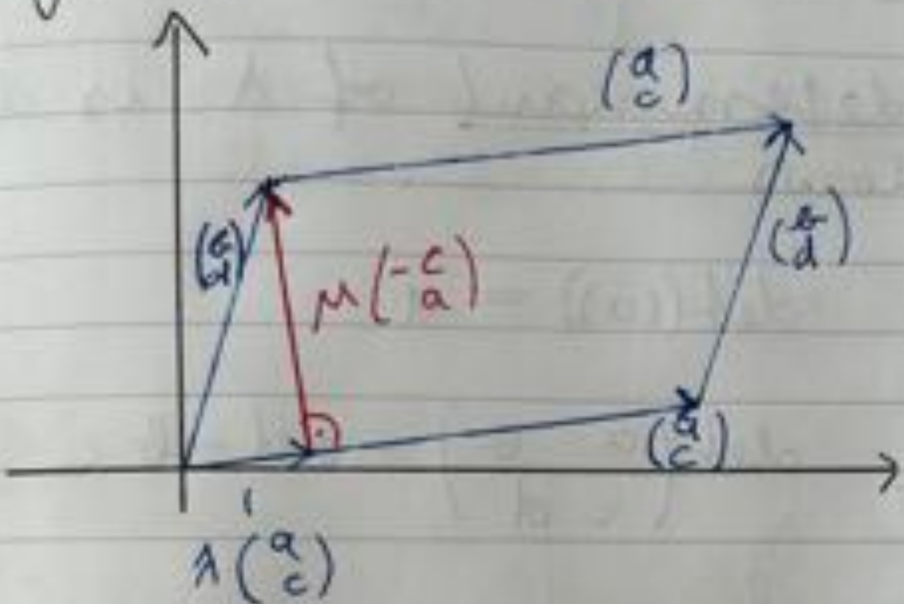
$$5 = -1+6 = (-1 \;\; 3)\circ\begin{pmatrix}1\\2\end{pmatrix}$$

$$= \lambda\,(-1 \;\; 3)\begin{pmatrix}3\\1\end{pmatrix} + \mu(-1 \;\; 3)\circ\begin{pmatrix}-1\\3\end{pmatrix}$$

$$= \lambda\cdot 0 + \mu\cdot 10$$

$\Rightarrow \mu = \frac{1}{2}$ and area $= 5$

(3) general



Suppose $\binom{a}{c} \neq \binom{0}{0}$.

$$\text{area} = \mu(a^2 + c^2).$$

$$= (-c, a) \circ \left(\lambda\binom{a}{c} + \mu\binom{-c}{a}\right)$$

$$= (-c\ a) \circ \binom{b}{d}$$

$$= \underline{\underline{ad - bc}}$$

We need a formula in higher

dimension.

Def 64: Let $A \in \mathbb{R}^{m\times m}$.

The determinant of A is defined as follows.

$m=1$: $\det((a)) := a$

$m=2$: $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc$

$m=3$: $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$:= a_{11} \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$- a_{21} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+ a_{31} \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$

We get $\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$ via:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$\lrcorner$

<u>$n>3$:</u> $\det(A) :=$

$$\sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

<u>Notation 65:</u> For $n \geq 2$ we also write $|A|$ for $\det(A)$, e.g.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

<u>Remark 66:</u> The signs follow the

matrix $\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \end{pmatrix}$

like a chess board.

<u>Example 67:</u>

$$\overbrace{\begin{vmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & -1 & 3 \\ -2 & 1 & 2 & 5 \\ 1 & 3 & -3 & 1 \end{vmatrix}}^{A}$$

$$= 1 \cdot \begin{vmatrix} 1 & -1 & 3 \\ 1 & 2 & 5 \\ 3 & -3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 & -1 \\ 1 & 2 & 5 \\ 3 & -3 & 1 \end{vmatrix}$$

$$+ (-2) \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 3 & -3 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= 1 \cdot M_{11} - 2 M_{21} + (-2) M_{31} - 1 M_{41}$$

↑
just notation

We need to compute the "minors" $M_{11}, M_{21}, M_{31}, M_{41}$

$$M_{11} = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 2 & 5 \\ 3 & -3 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 5 \\ -3 & 1 \end{vmatrix} - 1 \begin{vmatrix} -1 & 3 \\ -3 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= (2-(-15)) - 1(-1-(-9)) + 3(-5-6)$$

$$= 17 - 8 - 33 = -24$$

$$M_{21} = \begin{vmatrix} 2 & 5 \\ -3 & 1 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ 2 & 5 \end{vmatrix}$$

$$= 17 - (-1) + 3(12) = 54$$

$$M_{31} = \begin{vmatrix} -1 & 3 \\ -3 & 1 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ -3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= 8 - (-1) + 3(5) = 24$$

$$M_{41} = \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix}$$

$$= -11 - 12 + 5 = -18$$

$$\det(A) = 1(-24) - 2(54) + (-2)(29) - 1(-18)$$

$$= \underline{\underline{-162}}.$$

<u>Question 68:</u> Suppose $A \in \mathbb{R}^{m \times m}$ has a zero-row, $A = \begin{pmatrix} & * & \\ 0 & \cdots & 0 \\ & * & \end{pmatrix}$.

What is $\det(A)$?

<u>Terminology 69:</u> $A \in \mathbb{R}^{m \times m}$.

$i, j \in \{1, \dots, m\}$.

(1) The (i,j)-submatrix $A^{(ij)}$ of A is defined by

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mm} \end{pmatrix}$$

(2) $M_{ij} := \det(A^{(ij)})$ is called (i,j) – minor of A

(3) $(-1)^{i+j} M_{ij}$ is called the (i,j)– cofactor of A.

(4) $\det(A) = \sum_{i=1}^{m} a_{i1} C_{i1}$ is called the <u>cofactor ex-pansion</u> wrt. the first column.

<u>Example 70:</u> $\begin{pmatrix} 1 & 1 & -7 \\ 1 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix} = A$

	$(2,1)$	$(2,3)$	$(2,2)$
$A^{(ij)}$	$\begin{pmatrix} 1 & -7 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & -7 \\ 2 & -1 \end{pmatrix}$
M_{ij}	6	0	12
C_{ij}	-6	-0	$+12$

Later we will see: $\det(A) = 1 \cdot (-6) + 3 \cdot (12) + 2(-0) = 30$

cofactor expansion wrt.
the 2^{nd} row.

Check: $\det(A) = 2\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & -7 \\ 1 & -1 \end{vmatrix}$

$$+2\begin{vmatrix} 1 & -7 \\ 3 & 2 \end{vmatrix} = 2(-5)-6 + 2(23)$$

$= 30$

Example 71: (Some basic examples)

(a) for diagonal matrices:

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix} \in \mathbb{R}^{m \times m}$$

$$\det(D) = \lambda_1 \begin{vmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_m \end{vmatrix} \mp 0 \begin{vmatrix} & \end{vmatrix} + 0 \begin{vmatrix} & \end{vmatrix}$$

$$-0 \begin{vmatrix} & \end{vmatrix} \pm \dots \pm 0 \begin{vmatrix} & \end{vmatrix}$$

$$= \lambda_1 \begin{vmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_m \end{vmatrix} = \lambda_1 \lambda_2 \begin{vmatrix} \lambda_3 & & \\ & \ddots & \\ & & \lambda_m \end{vmatrix}$$

$$= \lambda_1 \lambda_2 \lambda_3 \cdot \ldots \cdot \lambda_m.$$

What about upper triangular matrices?

$$T = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & 0 & \ddots \\ & & & \lambda_m \end{pmatrix}$$

$$\det(T) = \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_m \quad ?$$

(b) $\det(I_m) = 1 \cdot \ldots \cdot 1 = 1.$

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(c) swapping successive rows.

<u>$m=2$:</u> $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -(bc - ad)$

$$= - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

n=3:

$$\begin{vmatrix} 2 & 3 & -2 \\ 1 & 3 & 5 \\ -1 & 2 & 6 \end{vmatrix} = 2\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ 2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 3 & 5 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & -2 \\ -1 & 2 & 6 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 2 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 5 \\ 3 & -2 \end{vmatrix}$$

$$\underset{(n=2\text{ case})}{=} -2\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} + \begin{vmatrix} 3 & -2 \\ 2 & 6 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -2 \\ 3 & 5 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 3 & -2 \\ 1 & 3 & 5 \\ -1 & 2 & 6 \end{vmatrix}$$

(d) For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\det\left(\begin{pmatrix} \lambda a & \lambda b \\ c & d \end{pmatrix}\right)$$

$$= \lambda(ad) - \lambda bc = \lambda(ad - bc)$$
$$= \lambda \det(A).$$

We collect the general statements in two propositions.

Proposition 72: (determinant under (E1))

Let $A = (a_{ij})_{ij} \in \mathbb{R}^{m \times m}$ and $\lambda \in \mathbb{R}$

and $i_0 \in \{1, 2, .., m\}$.

Then $\det \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \lambda a_{i_0 1} & \cdots & \lambda a_{i_0 m} \\ \vdots & & \vdots \\ a_{m1} & & a_{mm} \end{pmatrix}$

$= \lambda \det(A)$

Proof: Exercise on Problem sheet 4.

Hint: (Induction on m)

base case: $m=1$: $\det((\lambda a_{11})) = \lambda a_{11} = \lambda \det((a_{11}))$

induction step $m>1$: Induction hypothesis (IH)

Suppose the statement is true for matrices with less than m rows.

To show: It is true for matrices with n rows.

$$~~dt~~ \det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \lambda a_{j1} & & \lambda a_{jn} \\ \vdots & & \\ a_{n1} & & a_{nn} \end{pmatrix}$$

$$= a_{11} \underbrace{\left| \begin{matrix} \\ \lambda a_{j2} \cdots \lambda a_{jn} \\ \\ \end{matrix} \right|}_{\text{we have (IH)}} - \cdots$$

$\square$

<u>Prop. 73:</u> (determinant under (E2))

For $A \in \mathbb{R}^{n \times n}$ $(n \geq 2)$ and $1 \leq j < k < n$ we have

$$\det(E_{jk}A) = -\det(A).$$

Proof: Given in the example class $\square$

Example 74:

(a) (two equal rows)

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 0 \\ 1 & 2 & 1 \end{vmatrix} \underset{\text{swap I and III.}}{\overset{}{=}} - \begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 0 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 2 \begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0 \underset{\cdot \frac{1}{2}}{\Longrightarrow} \begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0.$$

(b) (zero row)

$$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 7 & 5 & 3 \end{vmatrix} \underset{\substack{\text{Prop 72} \\ \text{with } \lambda = 0 \\ \text{and } i_0 = 2}}{=} 0 \begin{vmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 7 & 5 & 3 \end{vmatrix} = 0$$

We still need to consider (E3).

<u>Prop 75</u> (additivity of det wrt. rows)

Suppose we are given matrices $A, B \in \mathbb{R}^{m\times m}$ of the form

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \cdots & a_{1m} \\ a_{21} & & a_{2m} \\ \vdots & & \vdots \\ a_{m1} & & a_{mm} \end{pmatrix}$$

Then $\det(D) = \det(A) + \det(B)$

for $D = \begin{pmatrix} a_{11}+b_{11} & a_{1m}+b_{1m} \\ a_{21} & a_{2m} \\ a_{m1} & a_{mm} \end{pmatrix}$.

<u>Proof:</u> <u>$m=1$:</u> $\det((a_{11}+b_{11}))$
(base case)

$= a_{11} + b_{11} = \det((a_{11})) + \det((b_{11}))$

<u>$m > 1$:</u> (IH): statement is true for matrices with less rows.
(induction step)

(IC): –"– for matrices with m rows.
induction claim

$$\det(\overset{D}{\cancel{A+B}}) = (a_{11}+b_{11})\, \overbrace{C_{11}}^{\text{Cofactor}}(\underset{D}{\cancel{A+B}})$$

$$+\, a_{21}\, C_{21}(\underset{D}{\cancel{A+B}}) + a_{31}\, C_{31}(\underset{D}{\cancel{A+B}})$$

$$+ \dots + a_{m1}\, C_{m1}(\underset{D}{\cancel{A+B}})$$

$$\underset{(IH)}{=} (a_{11}+b_{11})\, C_{11}(A) + \sum_{j=1}^{m} a_{j1}\left(C_{j1}(A) + C_{j1}(B)\right)$$

$$\underset{C_{11}(A) = C_{11}(B)}{=} \sum_{j=2}^{m} a_{j1}\left(C_{j1}(A) + C_{j1}(B)\right) + a_{11}C_{11}(A) + b_{11}C_{11}(B)$$

$$= \det(A) + \det(B). \qquad \square$$

Question 76: How to obtain additivity for other rows?

Terminology 77: 72 and 75 mean that det is multi-linear in rows

Prop. 78: (determinant under (E3))

(E3) does not change the determinant.

Proof: $1 \leq j \neq k \leq m$, $A \in \mathbb{R}^{m \times m}$, $\lambda \in \mathbb{R}$

$$\det \begin{pmatrix} a_{j1} + \lambda a_{k1} & \cdots & a_{jm} + \lambda a_{km} \\ a_{k1} & & a_{km} \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ \text{Prop. 75}}}{=} \det(A) + \det \begin{pmatrix} \lambda a_{k1} & \cdots & \lambda a_{km} \\ a_{k1} & \cdots & a_{km} \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ \text{Prop 72}}}{=} \det(A) + \lambda \det \begin{pmatrix} a_{r1} & \cdots & a_{rm} \\ a_{r1} & & a_{rm} \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ 74(b)}}{=} \det(A) + \lambda 0 = \det(A) \qquad \square$$

Example 79: (See Example 67)

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 2 & 1 & -1 & 3 \\ -2 & 1 & 2 & 5 \\ 1 & 3 & -3 & 1 \end{vmatrix} \underset{(E3)}{=} \begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & -5 & 5 \\ 0 & 3 & 6 & 3 \\ 0 & 2 & -5 & 2 \end{vmatrix}$$

$$\underset{\substack{\uparrow \\ \text{cofactor exp.} \\ 1^{st} \text{ column (by} \\ \text{definition)}}}{=} \begin{vmatrix} -1 & -5 & 5 \\ 3 & 6 & 3 \\ 2 & -5 & 2 \end{vmatrix}$$

$$\underset{(E3)}{=} \begin{vmatrix} -1 & -5 & 5 \\ 0 & -9 & 18 \\ 0 & -15 & 12 \end{vmatrix} = - \begin{vmatrix} -9 & 18 \\ -15 & 12 \end{vmatrix}$$

$$\underset{(E1)}{=} (-1)(-9)(3) \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 27 \cdot (-6) = -162.$$

Example 80: (formulas for determinant of small matrices)

(a) $\overset{+}{}\begin{vmatrix} a & b \\ c & d \end{vmatrix}\overset{-}{} = +ad - bc$

(b) $\left|\begin{array}{ccc|cc} a & b & c & a & b \\ e & f & g & e & f \\ h & i & j & h & i \end{array}\right.$

$$= +afj + bgh + cei - hfc - iga - jeb$$

Example: $\begin{vmatrix} 1 & 2 & 1 \\ -1 & -2 & 1 \\ 2 & 3 & -5 \end{vmatrix}$

$$= +10 + 4 + (-3) - (-4) - 3 - 10$$

$$= 2$$

Remark 81: Let $A \in \mathbb{R}^{m\times n}$

(a) $\exists$ a diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$ with $\lambda_1, \dots, \lambda_m \neq 0$
$\exists$ $E \in \mathbb{R}^{m\times m}$ a product of elementary addition matrices

such that

$$EA = DR$$

where R is the reduced row echelon form of A.

Example: $A = \begin{pmatrix} 0 & -1 & 3 \\ 2 & 1 & 4 \\ 1 & -1 & 1 \end{pmatrix}$

$$\left(A \,\middle|\, I_3 \right) \xrightarrow{I+II} \left(\begin{array}{ccc|ccc} 2 & 0 & 7 & 1 & 1 & \\ 2 & 1 & 4 & & 1 & \\ 1 & -1 & 1 & & & 1 \end{array}\right)$$

$$\xrightarrow{II-I} \left(\begin{array}{ccc|ccc} 2 & 0 & 7 & 1 & 1 & \\ 0 & 1 & -3 & -1 & & \\ 1 & -1 & 1 & & & 1 \end{array}\right) \xrightarrow{III-\frac{1}{2}I} \left(\begin{array}{ccc|ccc} 2 & 0 & 7 & 1 & 1 & \\ 0 & 1 & -3 & -1 & & \\ 0 & -1 & -\frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{III}+\mathrm{II}} \left(\begin{array}{ccc|ccc} 2 & 0 & 7 & 1 & 1 & \\ 0 & 1 & -3 & -1 & & \\ 0 & 0 & -\frac{11}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{II}-\frac{6}{11}\mathrm{III}} \left(\begin{array}{ccc|ccc} 2 & 0 & 7 & 1 & 1 & \\ & 1 & 0 & -\frac{2}{11} & +\frac{3}{11} & -\frac{6}{11} \\ & & -\frac{11}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right)$$

$$\xrightarrow{\mathrm{I}+\frac{14}{11}\mathrm{III}} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -\frac{10}{11} & \frac{4}{11} & \frac{14}{11} \\ & 1 & & -\frac{2}{11} & \frac{3}{11} & -\frac{6}{11} \\ & & -\frac{11}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} -\frac{10}{11} & \frac{4}{11} & \frac{14}{11} \\ -\frac{2}{11} & \frac{3}{11} & -\frac{6}{11} \\ -\frac{3}{2} & -\frac{1}{2} & 1 \end{pmatrix}}_{E} A = \underbrace{\begin{pmatrix} 2 & & \\ & 1 & \\ & & -\frac{11}{2} \end{pmatrix}}_{D.} R$$

$R = I_3.$

(b) $\underset{\mathbb{R}^{m\times m}}{\underset{\cap}{A}}$ is invertible $\Leftrightarrow R = I_m$

Proof: "$\Leftarrow$" Theorem 44 (b).
because $D^{-1}EA = R = I_m$

"$\Rightarrow$" If A is invertible then R has no zero-row by Theorem 46 ($1^\circ \Rightarrow 5^\circ$)

$$\underset{\substack{\uparrow \\ R \text{ reduced row echelon form}}}{\Rightarrow} \qquad R = I_m \qquad \square$$

(c) $\underset{A \in \mathbb{R}^{m\times m}}{\det(A)} \underset{\substack{\uparrow \\ (E3) \\ \text{Prop 78}}}{=} \det(EA) = \det(DR)$

$$\underset{\substack{\uparrow \\ (E1) \\ \text{Prop 72}}}{=} \lambda_1 \cdots \lambda_m \det(R)$$

$$= \begin{cases} \lambda_1 \cdots \lambda_m, & A \text{ invertible because } R = I_m \\ 0, & A \text{ non-invertible because } R \text{ has a zero-row.} \end{cases}$$

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<u>Theorem 82:</u> Given $A \in \mathbb{R}^{n \times n}$.

Then A is invertible iff $\det(A) \neq 0$

<u>Proof</u> Remark 81 (c) $\square$

<u>Example 83:</u> For which

$b \in \mathbb{R}$ is $A_b := \begin{pmatrix} 2 & -1 & 3 \\ 1 & 1 & -1 \\ 0 & 1 & b \end{pmatrix}$

invertible?

$$\det(A_b) = 2b + 3 - (-2) - (-b)$$

$$= 5 + 3b$$

$$\Rightarrow (\det(A_b) = 0 \Leftrightarrow b = -\frac{5}{3})$$

Answer: A_b is invertible iff

$b \neq -\frac{5}{3}$.

Theorem 83 (main properties of the determinant)

(i) Let $A, B \in \mathbb{R}^{m\times m}$. Then

(i1) $\det(AB) = \det(A)\det(B)$
(det is multiplicative)

(i2) $\det(A^T) = \det(A)$

(ii) Prop. 72, 73 and 78 remain true for column operations.

(iii) For every row and every column the cofactor expansion gives the determinant.

Example 84: (a) $\begin{vmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ 0 & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{vmatrix} \underset{\substack{\text{cofactor}\\\text{expansion}\\1^{st}\text{ row.}}}{\overset{\uparrow}{=}} (-1)^{1+m} \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix}$

$= (-1)^{m+1}$

(b) Take $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{\ell\times\ell}$ and $C \in \mathbb{R}^{m\times\ell}$

$$\begin{vmatrix} A & C \\ & B \end{vmatrix} = \left| \begin{pmatrix} I_m & \\ & B \end{pmatrix} \begin{pmatrix} I_m & C \\ & I_\ell \end{pmatrix} \begin{pmatrix} A & \\ & I_\ell \end{pmatrix} \right|$$

$$\underset{\uparrow}{=} \begin{vmatrix} I_m & \\ & B \end{vmatrix} \begin{vmatrix} I_m & C \\ & I_\ell \end{vmatrix} \begin{vmatrix} A & \\ & I_\ell \end{vmatrix}$$

det is multi-
plicative.

$$\underset{\uparrow}{=} |B| \cdot 1 \; |A| = |A||B|$$

- cofactor expansion
  - 1^{st} column m-times
  - last column m-times
- det of upper triangular matrix

(c)

$$\det \begin{pmatrix} 1 & 3 & 7 & 22 \\ 2 & 9 & 3 & 51 \\ & & 2 & 5 \\ & & 1 & 1 \end{pmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix} \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3(-3)$$

$$= -9$$

Proof: (i) (i1) We have AB is invertible iff A and B are invertible.
(if: Lemma 41(b),
only if: Problem sheet 3)

Case AB is not invertible:

Theorem 82 $\Rightarrow \det(AB) = 0 = \det(A)\det(B)$

Case: AB is invertible: Rk 81 $\Rightarrow EA = D$.

$$\Rightarrow \det(AB) \underset{(E3)}{=} \det(EAB) = \det(DB)$$

$$\underset{(E1)}{=} \lambda_1 \cdots \lambda_m \det(B)$$

$$\underset{81(c)}{=} \det(A)\det(B).$$

(i2) A invertible $\Leftrightarrow A^T$ is invertible.
Case A is not invertible:
Thm 82 $\Rightarrow \det(A) = 0 = \det(A^T)$

Case A is invertible: Rk 81 $\Rightarrow EA = D$

$$\Rightarrow A^T E^T = D^T$$

and $\det(A^T) = \det(A^T)\cdot 1$

$\underset{\substack{\uparrow \\ E^T = E_1^T \cdots E_\ell^T \\ E_i \text{ el. add. matrices} \\ \text{all } \det(E_i^T)=1}}{=} \det(A^T)\det(E_1^T)\cdots\det(E_\ell^T)$

$\underset{(i1)}{=} \det(A^T E^T) = \det(D^T) \underset{\substack{\uparrow \\ 71(a)}}{=} \lambda_1 \cdots \lambda_m$

$\underset{\substack{\uparrow \\ \text{Rk } 81(c)}}{=} \det(A).$

(ii) and (iii) follow from (i2) and Prop. 73 □

Example 85:

$$\begin{vmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{vmatrix} \underset{\substack{\uparrow \\ \text{last column}}}{=} (-1)^{m+1} \begin{vmatrix} & \overbrace{\quad}^{m-1} & 1 \\ & \iddots & \\ 1 & & \end{vmatrix} = (-1)^{m+1}(-1)^{m-1+1} \begin{vmatrix} & 1 \\ 1 & \end{vmatrix}_{m-1}$$

$$= (-1)^{(m+1)+m+(m-1)+\dots+2}$$

$$= (-1)^{\frac{(m+3)m}{2}}$$

Summary 85: $A, B \in \mathbb{R}^{n \times m}$

Operation	det
(E1) (rows, column)	$\det(E_j(\lambda) A) = \det(A E_j(\lambda)) = \lambda \det(A)$
(E2)	$\det(E_{jk} A) = \det(A E_{jk}) = -\det(A)$
(E3)	$\det(E_{jk}(\mu) A) = \det(A E_{jk}(\mu)) = \det(A)$
product	$\det(AB) = \det(A)\det(B)$
sum of two rows/columns	$\det\begin{pmatrix} C_1 \\ a_1+b_1, \dots, a_m+b_m \\ C_2 \end{pmatrix}$ $\det\begin{pmatrix} C_1 \\ a_1, \dots, a_m \\ C_2 \end{pmatrix} + \det\begin{pmatrix} C_1 \\ b_1, \dots, b_m \\ C_2 \end{pmatrix}$ similar for column additivity.
transpose	$\det(A^T) = \det A$
inverse	if A is invertible: $\det(A^{-1}) = \frac{1}{\det(A)}$

Date

Example 87

$$\begin{vmatrix} 2 & 3 & 1 & 0 \\ 1 & 2 & 7 & 1 \\ 2 & -5 & 3 & 0 \\ 1 & -10 & 2 & 5 \end{vmatrix}$$

$$\underset{\text{last column}}{\overset{=}{\uparrow}} (-1)^{2+4} \cdot 1 \begin{vmatrix} 2 & 3 & 1 \\ 2 & -5 & 3 \\ 1 & -10 & 2 \end{vmatrix} + (-1)^{4+4} \cdot 5 \begin{vmatrix} 2 & 3 & \dots \\ 1 & 2 & \dots \\ 2 & -5 & \dots \end{vmatrix}$$

$$\underset{(E3)}{=} \begin{vmatrix} 0 & 23 & -3 \\ 0 & 15 & -1 \\ 1 & -10 & 2 \end{vmatrix} + 5 \begin{vmatrix} 0 & -1 & -13 \\ 1 & 2 & 7 \\ 0 & -9 & -11 \end{vmatrix}$$

$$\underset{1^{st}\text{ column}}{\overset{=}{\uparrow}} + (-23-(-45)) + 5\,(-1)\,1\,(\cancel{22}-11 \dots$$

$$= 22 + \overset{530}{\cancel{475}} = \cancel{497}.\ 552$$

We can write the cofactor expansions using matrix multiplication

Def 88: Let $A \in \mathbb{R}^{m \times m}$. We define

- $\mathrm{Minor}(A) := \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1m} \\ M_{21} & M_{22} & & M_{2m} \\ \vdots & & & \vdots \\ M_{m1} & M_{m2} & & M_{mm} \end{pmatrix}$

  "minor matrix of A"

- $\mathrm{Cof}(A) := \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ C_{21} & C_{22} & & C_{2m} \\ \vdots & & & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{pmatrix}$

  "cofactor matrix of A"

  (recall: $C_{ij} = (-1)^{i+j} M_{ij}$)

- $\mathrm{Adj}(A) := \mathrm{Cof}(A)^T$

  "adjoint of A"

Example 89:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 13 & 1 & 2 \end{pmatrix}$$

$$\text{Minor}(A) = \begin{pmatrix} 11 & 19 & -62 \\ 6 & 4 & -37 \\ -3 & -2 & 1 \end{pmatrix}$$

$$\text{Cof}(A) = \begin{pmatrix} 11 & -19 & -62 \\ -6 & 4 & 37 \\ -3 & 2 & 1 \end{pmatrix}$$

$$\text{Adj}(A) = \begin{pmatrix} 11 & -6 & -3 \\ -19 & 4 & 2 \\ -62 & 37 & 1 \end{pmatrix}$$

Multiply with A:

$$\text{Adj}(A)\,A = \begin{pmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{pmatrix}$$

$$= \det(A) \cdot I_3$$

$$|A| = -(-1)(2-39) + 2(10-9) = -35$$

Theorem 90: For $A \in \mathbb{R}^{m \times m}$ we have

$$\mathrm{Adj}(A)\,A = A\,\mathrm{Adj}(A) = \det(A)\, I_m .$$

Proof: We just prove $A \cdot \mathrm{Adj}(A) = \det(A)\, I_m$

The other equation is similar.

$$A\,\mathrm{Adj}(A) =: (b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le m}}.$$

For $i = j$: $b_{ii} = \sum_{k=1}^{m} a_{ik} C_{ik} \underset{\substack{\uparrow \\ \text{Thm } 83 \text{ (iii)}}}{=} \det(A)$

For $i \neq j$: $b_{ij} = \sum_{k=1}^{m} a_{ik} C_{jk}$

$$= \begin{array}{c} \\ i \\ j \\ \\ \end{array} \begin{vmatrix} a_{11} & \cdots & a_{1m} \\ a_{i1} & \cdots & a_{im} \\ a_{i1} & \cdots & a_{im} \\ a_{m1} & \cdots & a_{mm} \end{vmatrix}$$

$$\overset{74(B)}{\underset{\downarrow}{=}} 0$$

Corollary 81: If $A \in \mathbb{R}^{m\times m}$ is invertible then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A).$$

Example 92: (a) (see Example 89)

$$\begin{pmatrix} 2 & 3 & 0 \\ 3 & 5 & -1 \\ 13 & 1 & 2 \end{pmatrix}^{-1} = \frac{1}{-35}\begin{pmatrix} 11 & -6 & -3 \\ -19 & 4 & 2 \\ -62 & 37 & 1 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\operatorname{Minor}(A) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

$$\operatorname{Cof}(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\operatorname{Adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{ad-bc} \operatorname{Adj}(A)$$

$$= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(c) Cor 91 gives a quick method to compute the inverse of a 3 by 3 matrix. (if invertible)

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -2 & 5 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\det(A) = (5 + 8 - 18) - (30 + 3 - 8)$$

$$= -5 - 25 = -30$$

$$\text{Minor}(A) = \begin{pmatrix} 2 & -4 & -16 \\ -5 & -5 & -5 \\ -11 & 7 & 13 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{-30} \begin{pmatrix} 2 & 5 & -11 \\ 4 & -5 & -7 \\ -16 & 5 & 13 \end{pmatrix} \quad \text{by Cor 91.}$$