LINEAR ALGEBRA 1
PROBLEM SHEET 4

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Problem 1 (10 points, conjugate matrices)**.** We call two square matrices $A,\ B \in \mathbb{R}^{m\times m}$ with m rows conjugate (to each other) if there is an invertible matrix C with m-rows such that

$$A = CBC^{-1}$$

(i) Suppose $A, B \in \mathbb{R}^{m\times m}$ are conjugate. Then $\mathrm{tr}(A) = \mathrm{tr}(B)$ and $\det(A) = \det(B)$.
(ii) Does the converse of (i) hold for 2×2-matrices, i.e. are two 2 by 2 matrices which share the trace and the determinant conjugate?

Problem 2 (20 points, computing determinants)**.** Compute the determinants of the following matrices.

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -3 & 1 & 5 \\ 1 & 2 & \frac{1}{2} & 6 & 1 \\ 0 & 1 & 2 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & b \end{pmatrix}.$$

For which entry b the last matrix is not invertible?

Problem 3 (10+5 points, definition of the determinant)**.** Let m be a positive integer and $\lambda_1, \ldots, \lambda_m$ be real numbers. We denote by $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ the diagonal matrix $D = (d_{ij})_{i,j} \in \mathbb{R}^{m\times m}$ with diagonal entries

$$d_{1,1} = \lambda_1,\ d_{2,2} = \lambda_2, \ldots,\ d_{m,m} = \lambda_m.$$

Given $A \in \mathbb{R}^{m\times m}$ prove the following assertions just using the definition of the determinant in Definition 64. (cofactor expansion with respect to the first column)

(i) Suppose there exists an index i_0, $1 \leqslant i_0 \leqslant m$, such that $\lambda_i = 1$ for all $i \neq i_0$. Then

$$\det(DA) = \lambda_{i_0} \det(A).$$

(ii) $\det(DA) = \lambda_1\lambda_2\lambda_3 \cdots \lambda_m \det(A)$.

Problem 4 (10+10, computing determinants)**.** Compute the following determinants:

(i)

$$\begin{vmatrix} 1 & 2 & 0 & 0 & 5 & 0 & -1 & 0 & 1 \\ 1 & 3 & 5 & 1 & -1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 2 \\ 1 & -1 & 0 & 0 & 3 & 0 & 1 & 0 & 1 \\ -2 & 1 & 3 & 4 & 1 & -2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 2 & 0 & 3 \\ -1 & 1 & 2 & 1 & 1 & -1 & 3 & 1 & 2 \\ 0 & 1 & -2 & 1 & 3 & 2 & 1 & 1 & 1 \end{vmatrix}.$$

See next page!

Date: Please hand in before the lecture by **1st of November 2023**. For all exercises the results need to be proven using results from this lecture and the lectures before, provided you give a reference. The intermediate steps for computations need to be provided.

(ii)

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_m^2 \\ \ddots & \ddots & & \ddots \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{vmatrix}.$$

for real numbers λ_i, $i = 1, \dots, m$ and $m \geqslant 2$.. (Hint: Perform a row operation with the last two rows.)