LinA I (Fall 2023)

Chapter 1

Systems of linear equations and matrices

1.1. Preliminaries in set theory.

Logic:

Def 1. A logical statement is a sentence for which we can decide whether it is true (1) or false (0).

p a l. statement. $t(p) = \begin{cases} 1, \text{true} \\ 0, \text{false} \end{cases}$

Examples 2:

a) $p =$ "Today is the 27^{th} of September"
$t(p) = 1.$

b) $q := \text{"}\exists x \in \mathbb{R} : x > 1\text{"}$
= "There exists a real number bigger than 1."

$t(q) = 1$

c) $r :=$ "All real numbers are greater than 1"

$t(r) = 0.$

<u>Def 2:</u> (logic operators)

a) "conjunction" (and, $\wedge$)

$\wedge$:

$p \backslash q$	0	1
0	0	0
1	0	1

$p \wedge q$

b) "disjunction" (or, $\vee$)

$\vee$:

$p \backslash q$	0	1
0	0	1
1	1	1

$p \vee q$

c) "negation" ($\neg$)

p	0	1
$\neg p$	1	0

d) "implication" (if p then q)

$$p \Rightarrow q :\!= q \vee \neg p$$

e) "equivalence" (p iff q)

$$p \Leftrightarrow q := (p \Rightarrow q) \wedge (q \Rightarrow p)$$

Exercise 3: Find the truth table for "$\Rightarrow$" and "$\Leftrightarrow$".

Example 4: p := "If there is a pink elephant in this room then Berlin is a city in China"

$$A(p) = ?$$

Example 5:

$$\underbrace{((p\wedge q)\vee(p\wedge r))}_{A}\Leftrightarrow\underbrace{(p\wedge(q\vee r))}_{B}$$

p	q	r	A	B
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

Thus $A \Leftrightarrow B$ is always true,
i.e. independent of p, q, r

"We call such a logical expression a <u>tautology</u>"

<u>Notation:</u>

quantifiers:
- $\forall$ "for all"
- $\exists$ "there exists"
- $\exists!$ "there exists exactly one"

<u>Def 6</u> (Set) A <u>set</u> is a collection of object from our imagination, visualization and nature The objects are called <u>elements</u>

~~Ex~~

We write $a \in A$ "a is an element of A".

<u>Example 7</u>: A ~~set~~ example for a set can be obtained as follows: Let $E(x)$ be a logic expression depending on x. It can be true or ~~talls~~ false depending on x

$$\{x \mid E(x) \text{ is true}\}$$

is a set.

<u>Ex</u>: a) $E(x) = $ " $\exists_{y \in \mathbb{R}} : y^2 = x$ "

$$\{x \mid E(x)\} = \mathbb{R}^{\geq 0} = [0, \infty[$$

b) $D(x) :=$ " $\exists a \in \mathbb{N} \; \exists b \in \mathbb{Z} :$

$$ax = b \text{"}$$

$\{x \mid D(x)\} = \mathbb{Q}$.
the set of rational numbers.

<u>Def 8</u>: Let $A_i, i \in I$, be sets

$\bigcap_{i \in I} A_i := \{x \mid \forall_{i \in I} : x \in A_i\}$

"intersection of $A_i, i \in I$"

$\bigcup_{i \in I} A_i := \{x \mid \exists_{i \in I} : x \in A_i\}$

"union of $A_i, i \in I$"

Let A, B be sets

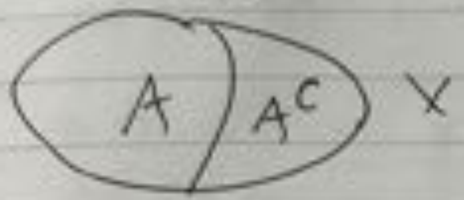
$A \setminus B := \{x \mid x \in A \wedge x \notin B\}$

"(set theoretic) difference of A and B"

a) Let X be a set. A set A is called a <u>subset</u> of X if for all $x \in A$ we have $x \in X$. We write "$A \subseteq X$".

c) Let X be a set and $A \subseteq X$

$A^c := X \setminus A$ "complement of A in X".



<u>Proposition 9:</u> (de Morgan's laws)

Let A, B, C be sets. Then

(a) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

(b) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

<u>Proof:</u> (a) We have to show "$\subseteq$" and "$\supseteq$"

"$\subseteq$": $x \in A \setminus (B \cap C)$

$\Rightarrow x \in A \wedge (x \notin B \text{ or } x \notin C)$

$\Rightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)$

Ex 5.

$\Rightarrow \; x \in A \setminus B$ or $x \in A \setminus C$

$\Rightarrow \; x \in (A \setminus B) \cup (A \setminus C)$.

"$\supseteq$" similar.

(b) Exercise. $\square$

Example 10:

$A = \{1, 2, 3, 4\}$ $B = \{2, 3\}$, $C = \{3, 5\}$

$A \setminus (B \cap C) = \{1, 2, 3, 4\} \setminus \{3\}$

$= \{1, 2, 4\}$

$A \setminus B = \{1, 4\}$
$A \setminus C = \{1, 2, 4\}$

$(A \setminus B) \cup (A \setminus C) = \{1, 2, 4\}$

Later we define maps.

1.2. Hyperplanes and linear systems.

For $n \in \mathbb{N} = \{1, 2, 3, 4, \dots\}$
we consider

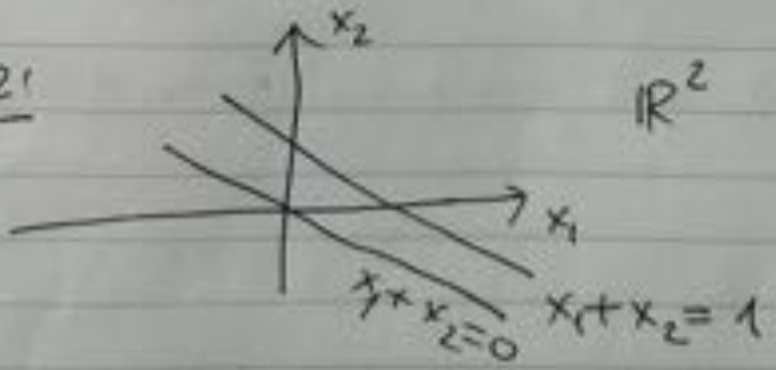
$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ times}} = \{(x_1, \dots, x_n) \mid x_1 \in \mathbb{R} \wedge \dots \wedge x_n \in \mathbb{R}\}$$

We have $\underline{0} = (0, \dots, 0) \in \mathbb{R}^n$ "zero point"

__Def 11:__ A subset H of $\mathbb{R}^n$
is called (a) an affine hyperplane in $\mathbb{R}^n$
if $\exists\, (a_1, \dots, a_n) \in \mathbb{R}^n \setminus \{\underline{0}\}$
$\exists\, c \in \mathbb{R}$: $H = \{\underline{x} \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i x_i = c\}$

(b) a linear hyperplane in $\mathbb{R}^n$
if H is an affine hyperplane
with $\underline{0} \in H$.

__Example 12:__

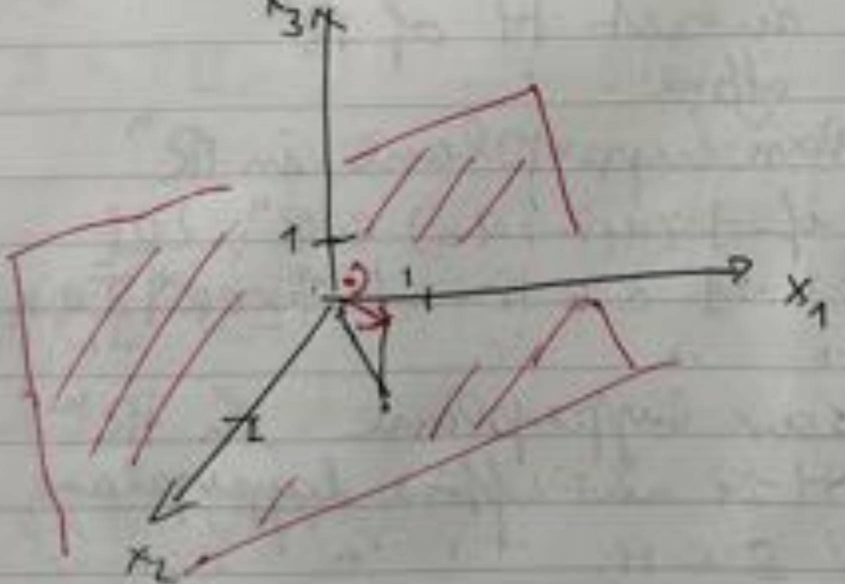


$H_1 = \{x \mid x_1 + x_2 = 1\}$ is an affine hyperplane in $\mathbb{R}^2$

$H_0 = \{x \mid x_1 + x_2 = 0\}$ is a linear hyperplane in $\mathbb{R}^2$

Example 13: (in $\mathbb{R}^3$)

$H = \{(x_1, x_2, x_3) \mid x_1 + 2x_2 + x_3 = 0\}$



Example 14: (optimization problem)

Company produces x tons of product A
y — " — B.

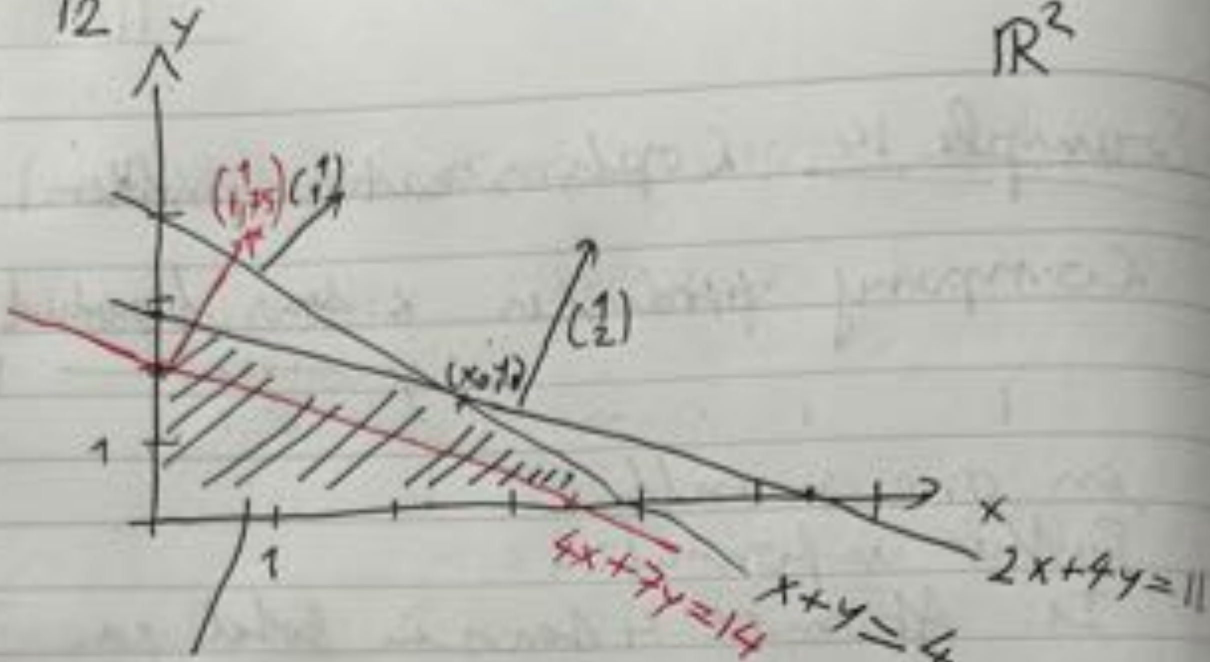
in one month.
Further information:
I1: At most 4 tons in total can be produced.
I2: production costs (in T€)

1 ton product A: 2
— " — B: 4
We can use only 11 T€.
I3: sale price (in T€)
1 ton A 4
1 ton B 7
I4: $x \geq 0$, $y \geq 0$.

Problem: Maximize the sales revenue.

$$sr(x,y) = 4x + 7y$$

$\mathbb{R}^2$



$$P := \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \wedge 0 \le y \wedge x+y \le 4 \wedge 2x+4y \le 11\}$$

. A production plan is a point in P.

We move the red line upwards to maximize $sr(x,y)$ on P.
The maximum value is taken at (x_0, y_0). the point in the intersection of

$$\ell_1 := \{(x,y) \in \mathbb{R}^2 \mid x+y = 4\} \text{ and}$$
$$\ell_2 := \{(x,y) \in \mathbb{R}^2 \mid 2x+4y = 11\},$$

i.e. $\ell_1 \cap \ell_2 = \{(x_0, y_0)\}$.

Question: How to find (x_0, y_0)?

We have to solve the "linear system"

in "matrix form"

$$\begin{array}{lc} \mathrm{I} & x + y = 4 \\ \mathrm{II} & 2x + 4y = 11 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 4 & 11 \end{array}\right)$$

$$\xrightarrow[\mathrm{II} - 2\mathrm{I}]{} \begin{array}{lc} \mathrm{I} & x + y = 4 \\ \mathrm{II} & 0 + 2y = 3 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 1 & 4 \\ & 2 & 3 \end{array}\right)$$

$$\xrightarrow[\cdot \frac{1}{2}\mathrm{II}]{} \begin{array}{lc} \mathrm{I} & x + y = 4 \\ \mathrm{II} & y = \frac{3}{2} \end{array} \qquad \left(\begin{array}{cc|c} 1 & 1 & 4 \\ & 1 & \frac{3}{2} \end{array}\right)$$

$$\xrightarrow[\mathrm{I} - \mathrm{II}]{} \begin{array}{lc} \mathrm{I} & x = \frac{5}{2} \\ \mathrm{II} & y = \frac{3}{2} \end{array} \qquad \left(\begin{array}{cc|c} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right)$$

We obtain $(x_0, y_0) = \left(\frac{5}{2}, \frac{3}{2}\right)$

and $\max\limits_{(x,y) \in P} \mathrm{Sr}(x,y) = \mathrm{Sr}(x_0, y_0)$

$$= \mathrm{Sr}\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= 4 \cdot \frac{5}{2} + 7 \cdot \frac{3}{2} = \frac{41}{2}$$

End of Lecture 1 (27.9)

Def 15:

(a) A linear equation in n-variables $x_1, \dots, x_n$ is an expression of the form

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b \quad (*)$$

where $a_1, a_2, \dots, a_n, b$ are constants, i.e. real numbers not depending on the variables.
A linear equation $(*)$ is called homogeneous if $b = 0$.

(b) A finite set of linear equations is called a linear system. The variables are called unknowns.

$$(**) \quad \begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_m \end{aligned}$$

It is called homogeneous linear

system if $b_1 = b_2 = \dots = b_m = 0$.

A solution of a linear system in n unknowns $x_1, \dots, x_n$ is a ~~tu~~ tuple $(s_1, \dots, s_n) \in \mathbb{R}^n$ such that after substituting

$$x_1 = s_1, \dots, x_n = s_n$$

all expressions in $(**)$ become true.

Remark 16: It is sometimes convinient to write a linear system in matrix form

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$$

$$= \left(A \mid b \right), \quad A \in \mathbb{R}^{m\times n}, \; b \in \mathbb{R}^{m\times 1}$$

It is called an augmented matrix, because the matrix A has been augmented by the vector b.

If a linear system is of a nice form then we can read the solutions.

<u>Example 17:</u> $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

The set of solutions is $\{(2-3s, s, 4) \mid s \in \mathbb{R}\}$.

If we write
" $x_1 = 2-3s$, $x_2 = s$, $x_3 = 4$, $s \in \mathbb{R}$"
then we call it the <u>general</u> ~~solo~~ <u>solution</u> of the linear system.

Def 18: A matrix $C \in \mathbb{R}^{m \times l}$ is called

(a) to be in <u>row echelon form</u>

if (re1) In every row the first non-zero entry is 1, and

(re2) for any two successive rows the leading 1 of the lower row appears further to the right than the leading 1 of the higher row.

(b) to be in <u>reduced row echelon form</u> if C is in ~~ex~~ row-echelon form and

(re3) for every leading 1 except of him only 0 occurs in its column

Example 19:

$$\begin{pmatrix} 1 & 7 & 3 & 2 & 4 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 \\ & & & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

row echelon form.

not reduce

How to get a reduced r e?

$$\xrightarrow[\mathrm{I} - 2\mathrm{II}]{} \begin{pmatrix} 1 & 7 & 3 & 0 & -2 & 0 & -8 & 24 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & -10 \\ 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 \\ 0 & \cdots & & & & & & 0 \\ 0 & \cdots & & & & & & 0 \end{pmatrix}$$

Def 20: The following operations are called <u>elementary row operations</u>:

(E1) multiplying a row with a non-zero scalar
$(\mathrm{I} \mapsto \lambda \mathrm{I})$

(E2) to swap two rows
$\begin{pmatrix} \mathrm{I} \\ \mathrm{II} \end{pmatrix} \to \begin{pmatrix} \mathrm{II} \\ \mathrm{I} \end{pmatrix}$

(E3) to add a multiple of a row to another one
$\mathrm{I} \longmapsto \mathrm{I} + \lambda \mathrm{II}$.

The Gauss-Jordan elimination is an algorithmus which transforms a matrix in two steps into reduced row echelon form using (E1), (E2), (E3):

Step 1: transforming the matrix into row echelon form "forward elimination"

Step 2: transforming the echelon form into a reduced echelon form "backwards elimination".

Example 21:

Step 1:

$$\begin{pmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ & 2 & -3 & 1 \end{pmatrix} \xrightarrow[(E2)]{\binom{II}{I}} \begin{pmatrix} 2 & -3 & 3 & 2 \\ -3 & -1 & 2 & 4 \\ & 2 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow[(E1)]{\frac{1}{2}I} \begin{pmatrix} 1 & -\frac{3}{2} & \frac{3}{2} & 1 \\ -3 & -1 & 2 & 4 \\ & 2 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow{II+3I} \begin{pmatrix} 1 & -\frac{3}{2} & \frac{3}{2} & 1 \\ 0 & -\frac{11}{2} & \frac{13}{2} & 7 \\ 0 & 2 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{2}{11}\mathrm{II}} \begin{pmatrix} 1 & -\frac{3}{2} & \frac{3}{2} & 1 \\ & 1 & -\frac{13}{11} & -\frac{14}{11} \\ & 2 & -3 & 1 \end{pmatrix}$$

$$\xrightarrow{\mathrm{III}-2\mathrm{II}} \begin{pmatrix} 0 & -\frac{3}{2} & \frac{3}{2} & 1 \\ & 1 & -\frac{13}{11} & -\frac{14}{11} \\ & & -\frac{7}{11} & \frac{39}{11} \end{pmatrix}$$

$$\xrightarrow{-\frac{11}{7}\mathrm{III}} \begin{pmatrix} 1 & -\frac{3}{2} & \frac{3}{2} & 1 \\ & 1 & -\frac{13}{11} & -\frac{14}{11} \\ & & 1 & -\frac{39}{7} \end{pmatrix}$$

in row echelon form

Step 2 :

$$\xrightarrow{\mathrm{I}+\frac{3}{2}\mathrm{II}} \begin{pmatrix} 1 & 0 & -\frac{3}{11} & -\frac{10}{11} \\ & 1 & -\frac{13}{11} & -\frac{14}{11} \\ & & 1 & -\frac{39}{7} \end{pmatrix}$$

$$\xrightarrow[\mathrm{I}+\frac{3}{11}\mathrm{III}]{\mathrm{II}+\frac{13}{11}\mathrm{III}} \begin{pmatrix} 1 & 0 & 0 & -\frac{187}{77} \\ & 1 & 0 & -\frac{605}{77} \\ & & 1 & -\frac{39}{7} \end{pmatrix} \begin{array}{l} \leftarrow \; -\frac{17}{7} \\ \leftarrow \; -\frac{55}{7} \\ \end{array}$$

How to check the result?

Think about it as a linear system

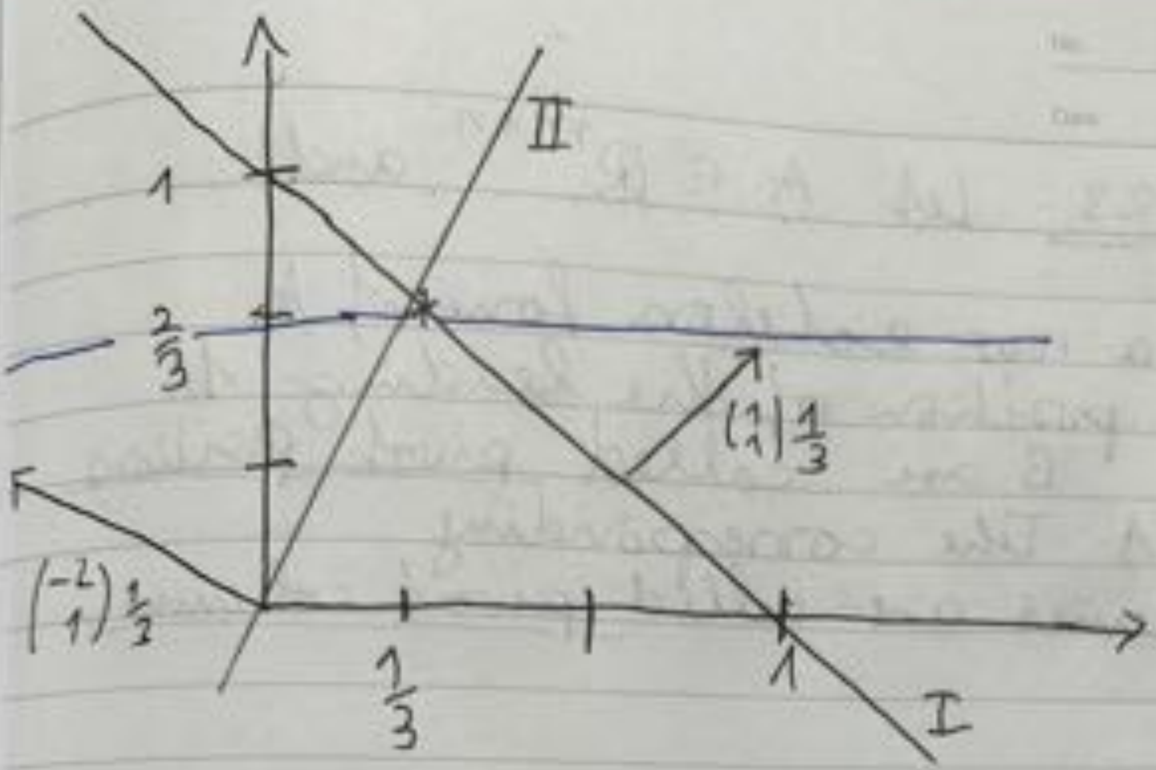
$$\left(\begin{array}{ccc|c} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ & 2 & -3 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & & & -17/7 \\ & 1 & & -55/7 \\ & & 1 & -39/7 \end{array}\right)$$

and check if the right side is a solution.

<u>Remark 22:</u> (Geometric interpretation of one step (E3) to obtain a zero entry)

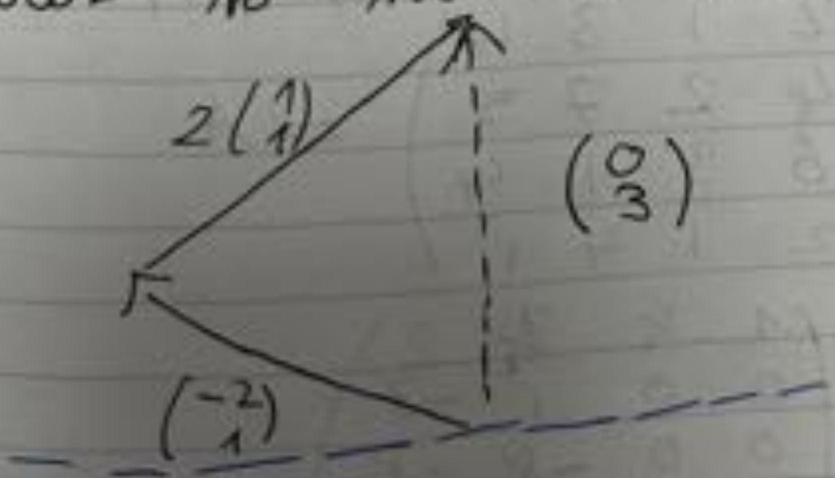
We consider $\mathbb{R}^2$.



$$\begin{aligned} \mathrm{I} \quad & x+y=1 \\ \mathrm{II} \quad & -2x+y=0 \end{aligned}$$

$$\xrightarrow{\mathrm{II}+2\mathrm{I}} \quad \begin{aligned} \mathrm{I} \quad & x+y=1 \\ \mathrm{II}^* \quad & 3y=0+2 \end{aligned}$$

Now II^* describes a line parallel to the x-axes.



Def 23: Let $A \in \mathbb{R}^{m \times n}$ and B a row echelon form of A. The position of the leading 1s in B are called <u>pivot position</u> of A. The corresponding columns are called <u>pivot columns</u>.

Example 24:

(a) $$\begin{pmatrix} 1 & 3 & 4 & 5 & -3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 7 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{is already in row echelon form}$$

pivot positions: $(1,1), (2,4), (3,5), (4,7)$

" columns: $1, 4, 5, 7$

(b) $$\begin{pmatrix} 2 & 1 & 3 & 4 \\ 4 & 2 & 7 & 5 \\ 6 & 3 & 1 & 4 \\ 2 & 1 & 4 & 1 \end{pmatrix} =: A$$

$$\longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -8 & -8 \\ 0 & 0 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -32 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & \cancel{-32}\,1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot positions: $(1,1), (2,3), (3,4)$
pivot columns: $1, 3, 4$.

(c) Geometric meaning of pivot positions:

In (b) it shows that we can realize the <u>intersection</u> of the 4 linear hyperplanes given by the rows of A & by a system of hyperplanes

H_1, H_2, H_3, H_4 where
H_2 is parallel to the x_1, x_2 axes
H_3 — " — x_1, x_2, x_3 axes.

End of Lecture 2 8/10/2023

There is one problem: Are pivot positions well-defined?

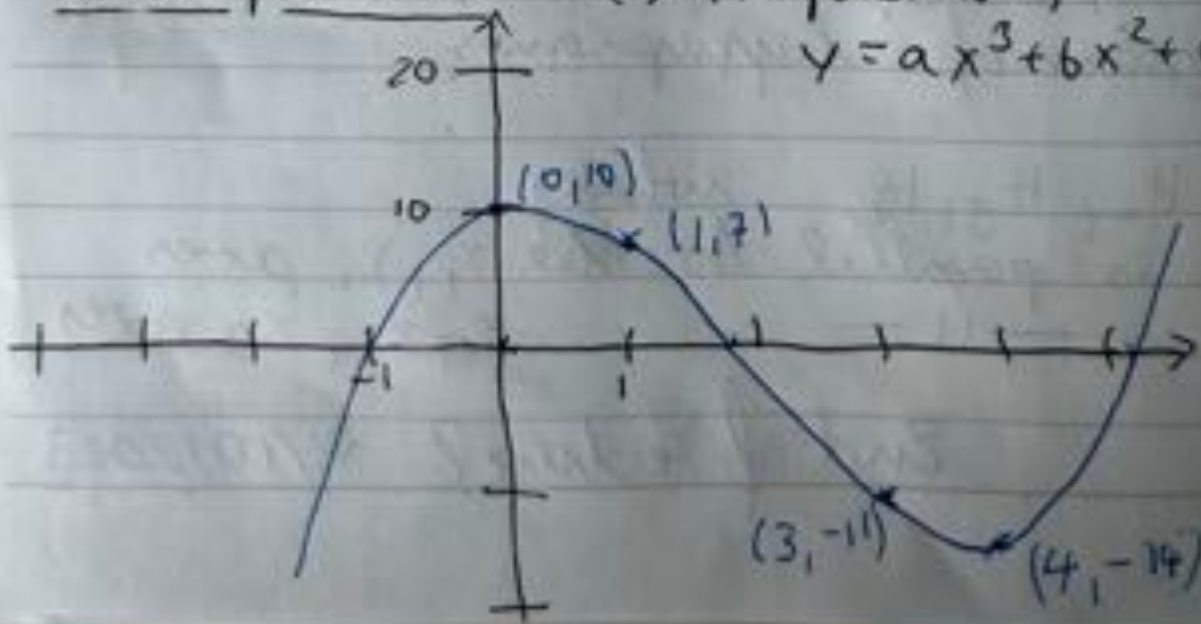
Theorem 25: Let $B, C \in \mathbb{R}^{m \times n}$ be reduced row echelon forms of $A \in \mathbb{R}^{n \times m}$.
Then $B = C$.

For the proof we need more theory. We come to the proof later.

Question 26: How does Theorem 25 imply the well-definedness of pivot positions?

Example 27: (interpolation)

$y = ax^3 + bx^2 + cx + d$



Find the coefficients.

Sol: We obtain the following linear system:

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 10 \\ 1 & 1 & 1 & 1 & 7 \\ 27 & 9 & 3 & 1 & -11 \\ 64 & 16 & 4 & 1 & -14 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -3 \\ 27 & 9 & 3 & 0 & -21 \\ 64 & 16 & 4 & 0 & -24 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -3 \\ 9 & 3 & 1 & 0 & -7 \\ 16 & 4 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -3 \\ 8 & 2 & 0 & 0 & -4 \\ 15 & 3 & 0 & 0 & -3 \\ 0 & 0 & & 1 & 10 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -3 \\ 4 & 1 & 0 & 0 & -2 \\ 5 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right)$$

$$\xrightarrow{\text{III} - \text{I}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & -3 \\ 4 & 1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -6 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 10 \end{array}\right)$$

<u>Sol^n</u>: $y = x^3 - 6x^2 + 2x + 10$

<u>Check</u>: Plug in the points
$(0, 10)$, $(1, 7)$, $(3, -11)$, $(4, -14)$.

1.3 Matrices and Matrix Operations

1.3.1 Interlude: Formal definition of matrixes.

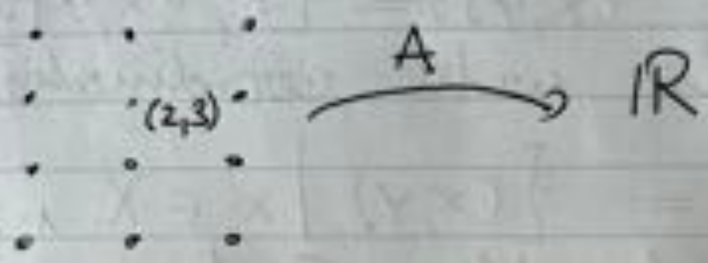
<u>Def 28:</u> Let X, Y be sets and $x \in X$ and $y \in Y$.

1) We call $(x,y) := \{x, \{x,y\}\}$ the pair with coordinates x and y.
2) $X \times Y := \{(x,y) \mid x \in X \wedge y \in Y\}$ is called the <u>Cartesian product</u> of X and Y.
3) A subset $F \subseteq X \times Y$ is called a <u>map</u> from X to Y if $F \neq \emptyset$ and
$$\forall x \in X \ \exists!\, y \in Y : (x,y) \in F.$$
We write $y = F(x)$, if $(x,y) \in F$, and $F : X \longrightarrow Y$.
4) Let $m, n \in \mathbb{N}$. We define
$$\mathbb{R}^{m \times n} := \{ f \mid f : \{1,\dots,m\} \times \{1,\dots,n\} \longrightarrow \mathbb{R} \}.$$
It is called the set of $m \times n$-matrices with real coefficients.

Example 29:

(a) For $x \in X, y \in Y$ with $x \neq y$ we have $(x,y) \neq (y,x)$.

(b) $m = 3$, $n = 4$, $A \in \mathbb{R}^{3\times 4}$



$$(i,j) \xmapsto{A} a_{ij}, \quad i \in \{1,\dots,\overset{3}{m}\}, \ j \in \{1,\dots,4\}$$

Remark 30: Two matrices $A \in \mathbb{R}^{m_1 \times n_1}$ and $B \in \mathbb{R}^{m_2 \times n_2}$ are equal if they are equal as maps

$$\{1,\dots,m_1\} \times \{1,\dots,n_1\} \xrightarrow{A} \mathbb{R}$$
$$\{1,\dots,m_2\} \times \{1,\dots,n_2\} \xrightarrow{B} \mathbb{R},$$

i.e. $m_1 = m_2 \wedge n_1 = n_2$ and $\forall_{1 \leq i \leq m_1} \forall_{1 \leq j \leq n_1} : a_{ij} = b_{ij}$.

13.2 Matrix operations

Look at the equation of a hyper plane:

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

The left hand side is a generalization of the multiplication in $\mathbb{R}$.

<u>Terminology 31:</u> An element

$v = (a_1, a_2, \dots, a_n) \in \mathbb{R}^{1\times n}$ is called a <u>row vector</u> and an element

$w = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^{m\times 1}$ is called a <u>column vector</u>

<u>Definition 32:</u> For $v = (a_1, \dots, a_m) \in \mathbb{R}^{1\times m}$ and $w = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^{m\times 1}$

we define a product

$$v \circ w := a_1b_1 + a_2b_2 + a_3b_3 + \dots + a_mb_m.$$

Example 33:

$$(1, \tfrac{1}{2}, 3, -1) \circ \begin{pmatrix} 2 \\ 1/3 \\ 7/5 \\ 1 \end{pmatrix} = 2 + \frac{1}{6} + \frac{21}{5} - \ldots$$

$$= \frac{1}{30}(60 + 5 + 126 - 30)$$

$$= \frac{161}{30}$$

Def. 34: (matrix multiplication)
The following map is called matrix multiplication:

$$\mathbb{R}^{m \times n} \times \mathbb{R}^{n \times l} \xrightarrow{\circ} \mathbb{R}^{m \times l}$$

defined via

$$A \circ B = C = (c_{ik})_{\substack{1 \le i \le m \\ 1 \le k \le l}}$$

with

$$c_{ik} := a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk}$$

Example 34:

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 3 & 1 \\ -1 & 2 & 2 \\ 5 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 9 & 2 \\ 16 & 2 & 13 \end{pmatrix}$$

The entry at $(2,2)$ is equal to

$$(2 \; 1 \; 3) \circ \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = 2$$

Example 35:

production line

month \ products	P_1	P_2	P_3	P_4
January	1	2	1	3
February	1,5	2	1	2,5
March	1	1,5	2	3

A

produced volume in tons.

Product \ T€/ton	product costs	sales price
P_1	1	2
P_2	2	3,5
P_3	1	1,5
P_4	0,5	1

B

We compute the month-(cost, sales revenue) matrix.

	Prod costs	sales revenue
Jan	7,5	13,5
Feb	7,75	14
March	7,5	13,25

$$C = A \circ B$$

month-gain matrix

	gain
Jan	6
Feb	6,25
March	5,75

$$D = C \circ \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We have another structure on $\mathbb{R}^{m\times n}$.

$$+ : \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$$

$$A + B := \left(a_{ij} + b_{ij}\right)_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}$$

called the addition of matrices, to be studied later. We denote

$$O := \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{m\times n}.$$

Addition + and multiplication have the following properties:

Proposition 35: Let $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times l}$ and $D \in \mathbb{R}^{l\times q}$.
Then we have:

(Ass $\circ$) $\quad A\circ(B\circ D) = (A\circ B)\circ D$
"associativity of $\circ$"

(L Dist $\circ,+$) $\quad A\circ(B+C) = (A\circ B)+(A\circ C)$
"left distributivity"

(RDist $\circ,+$) $(B+C)\circ D = (B\circ D)+(C\circ D)$
"right distributivity of $\circ,+$"

(LUnit) $I_m \circ A = A$ for $I_m = ($
$\in \mathbb{R}^{m\times m}$

I_m is called the <u>left unit</u> for $\mathbb{R}^{m\times n}$.

(RUnit) $A\circ I_n = A$
<u>right unit</u> for $\mathbb{R}^{m\times n}$.

<u>Proof:</u> We prove (Ass $\circ$). The remaining is left as an exercise.

$A = (a_{ij})$, $B = (b_{jk})$, $D = (d_{ks})$

Put $E = (e_{is})_{\substack{1\le i\le m\\ 1\le s\le q}} := A\circ(B\circ D)$

and $F := (f_{is}) := (A\circ B)\circ D$.

<u>To show:</u> $\forall_{1\le i\le m}\forall_{1\le s\le q}!\ e_{is} = f_{is}$.

Take $1 \le i \le m$ and $1 \le s \le s$.
Then

$$e_{is} = \sum_{j=1}^{n} a_{ij} (B \circ D)_{js}$$

$$= \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{\ell} b_{j\ell} d_{\ell s} \right)$$

$$\underset{\text{(L Dist in } \mathbb{R})}{=} \sum_{j=1}^{n} \sum_{k=1}^{\ell} a_{ij} (b_{j\ell} d_{\ell s})$$

$$\underset{(\text{Ass } \cdot_{\mathbb{R}})}{=} \sum_{j=1}^{n} \sum_{k=1}^{\ell} (a_{ij} b_{j\ell}) d_{\ell s}$$

$$\underset{(\text{Commutativity of } +_{\mathbb{R}})}{=} \sum_{k=1}^{\ell} \sum_{j=1}^{n} (a_{ij} b_{j\ell}) d_{\ell s}$$

$$\underset{(\text{R Dist } \cdot_{\mathbb{R}})}{=} \sum_{k=1}^{\ell} \left(\sum_{j=1}^{n} (a_{ij} b_{j\ell}) \right) d_{\ell s}$$

$$= \sum_{k=1}^{\ell} (A \cdot B)_{i\ell} d_{\ell s} = f_{is}.$$

$\square$

Example 36: (a) (for $m = n = l = q =$

$$E := \left(\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 5 & 13 \end{pmatrix}$$

$$F := \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 2-3 \\ 5 & 4+9 \end{pmatrix} = E$$

(b) Compare with Example 34.

End of Lecture 11.-10-201

Application 37: Use matrix multiplication to understand Gauss-Jordan elimination.

Take
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1/3 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

cf Sec Def 20:

For (E1) $\quad I \longrightarrow \lambda I \quad (\lambda \in \mathbb{R}\setminus\{0\})$

$$A \longrightarrow \begin{pmatrix} \lambda & & \\ & 1 & \\ & & 1 \end{pmatrix} A = \begin{pmatrix} \lambda & 2\lambda & -\lambda \\ 2 & \frac{1}{3} & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

(E2) $\quad \begin{pmatrix} I \\ II \end{pmatrix} \longrightarrow \begin{pmatrix} II \\ I \end{pmatrix}$

$$A \longrightarrow \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} A = \begin{pmatrix} 2 & \frac{1}{3} & 1 \\ 1 & 2 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

(E3) $\quad I \longrightarrow I + \lambda II \quad (\lambda \in \mathbb{R})$

$$A \longrightarrow \begin{pmatrix} 1 & \lambda & \\ & 1 & \\ & & 1 \end{pmatrix} A = \begin{pmatrix} 1+2\lambda & 2+\frac{\lambda}{3} & -1+\lambda \\ 2 & \frac{1}{3} & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

Def 38: Let $1 \leq i \leq m$, $1 \leq j < k \leq m$, $\lambda \in \mathbb{R}\setminus\{0\}$ and $\mu \in \mathbb{R}$.
We put

$$E_i(\lambda) := \begin{array}{c} \scriptstyle (i) \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \end{array} \in \mathbb{R}^{m\times m}$$

"elementary multiplication matrix"

(j,k)

$$E_{jk} := \begin{array}{r} \\ \\ j \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

(k,j)

"elementary permutation matrix"

(j,k)

$$E_{jk}(\mu) = \begin{array}{r} \\ \\ j \\ \\ k \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & \mu & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$$E_{kj}(\mu) = \begin{array}{r|l} & \begin{array}{ccccccc} & & j & & & & \end{array} \\ \begin{array}{r} \\ \\ j \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & \mu & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

(k,j)

"elementary addition matrix".

The matrices $E_i(\lambda)$, E_{jk}, $E_{jk}(\mu)$, and $E_{kj}(\mu)$ are called <u>elementary matrices</u>

<u>Example 39:</u> rre form of

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & \frac{1}{3} & 1 \\ 0 & 2 & 0 \end{pmatrix} \qquad \Bigg| \qquad \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

(memorise full operation)

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{pmatrix} = E_1 \longrightarrow \qquad \begin{pmatrix} 1 & 2 & -1 \\ 2 & \frac{1}{3} & 1 \\ & 1 & \end{pmatrix} \qquad \Bigg| \qquad \begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{pmatrix} = E_2 \longrightarrow \qquad \begin{pmatrix} 1 & 2 & -1 \\ 0 & -\frac{11}{3} & 3 \\ 0 & 1 & 0 \end{pmatrix} \qquad \Bigg| \qquad \begin{pmatrix} 1 & & \\ -2 & 1 & \\ & & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} = E_3 \longrightarrow \qquad \begin{pmatrix} 1 & 2 & -1 \\ & 1 & \\ & -\frac{11}{3} & 3 \end{pmatrix} \qquad \Bigg| \qquad \begin{pmatrix} 1 & & \\ 0 & 0 & \frac{1}{2} \\ -2 & 1 & \end{pmatrix}$$

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$$\xrightarrow{E_4 = \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{4}{3} & 1 \end{pmatrix}} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & & \\ & 1 & & 0 & 0 & \frac{1}{2} \\ & 0 & 3 & -2 & 1 & \frac{4}{6} \end{array}\right)$$

$$\xrightarrow{E_5 = \begin{pmatrix} 1 & -2 & \\ & 1 & \\ & & 1 \end{pmatrix}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ & 1 & 0 & 0 & 0 & \frac{1}{2} \\ & & 3 & -2 & 1 & \frac{4}{6} \end{array}\right)$$

$$\xrightarrow{E_6 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{3} \end{pmatrix}} \left(\begin{array}{ccc|ccc} 1 & & -1 & 1 & 0 & -1 \\ & 1 & & 0 & 0 & \frac{1}{2} \\ & & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{4}{18} \end{array}\right)$$

$$\xrightarrow{E_7 = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}} \left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & 1 & \end{array}\right. \underbrace{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{7}{18} \\ 0 & 0 & \frac{1}{2} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{18} \end{pmatrix}}_{= P}$$

Then $P = E_7 E_6 E_5 \cdots E_2 E_1$

and $PA = I_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

Check AP. We get I_3.
This is not a coincident.

Def 40: Let $A \in \mathbb{R}^{m \times m}$. We call A <u>invertible</u> if $\exists\, B \in \mathbb{R}^{m \times m}$ such that

$$BA = AB = I_m.$$

Lemma 41: (a) Let $A \in \mathbb{R}^{m \times m}$ be invertible and $B, C \in \mathbb{R}^{m \times m}$ such that

$$BA = AB = I_m$$

and

$$CA = AC = I_m.$$

Then $B = C$. Write $A^{-1} := B$. "The inverse of A".

(b) Let $A_1, A_2 \in \mathbb{R}^{m \times m}$ be invertible. Then $A_1 \circ A_2$ is invertible and

$$(A_1 \circ A_2)^{-1} = A_2^{-1} \circ A_1^{-1}$$

(c) All elementary matrices are invertible.

<u>Proof:</u> (a) $B = B\, I_m = B(AC)$

$$\underset{\text{(Ass o)}}{=} (BA)C = I_m C = C.$$

(b) $(A_2^{-1} \cdot A_1^{-1})(A_1 A_2)$

$= A_2^{-1}((A_1^{-1} A_1)\, A_2)$

$= A_2^{-1} I_m A_2 = I_m$

$(A_1 A_2)(A_2^{-1} A_1^{-1}) = I_m$ similar

(c) $E_i(\lambda)^{-1} = E_i(\lambda^{-1})$

$E_{jk}^{-1} = E_{jk}$

$E_{jk}(\mu)^{-1} = E_{jk}(-\mu)$

$E_{kj}(\mu)^{-1} = E_{kj}(-\mu)$ $\square$

Def 42: Two matrices $A, B \in \mathbb{R}^{m \times n}$ are called row-equivalent if we can obtain B from A by elementary row-operations.

Example 43:

(a) $A \in \mathbb{R}^{m \times n}$. Then A is row-equivalent to any of its row echelon forms.

(b) Suppose $A, B \in \mathbb{R}^{m \times m}$ are row-equivalent.

Then: A is invertible iff B is invertible.

Proof: $A = EB \qquad E = E_s E_{s-1} \cdots E_1$

(E_j elementary).

Lemma 41 (c) and (*) $\Rightarrow$ E is invertible.

"$\Rightarrow$" Suppose A is invertible.

$\overset{\Rightarrow}{\underset{E^{-1} \text{ is invertible}}{\uparrow}}$ A and E^{-1} are inve[rtible]

$\overset{\Rightarrow}{\underset{\text{Lemma 41 (b)}}{\uparrow}}$ $B = E^{-1}A$ is invertible

"$\Leftarrow$" similar $\square$

(c) $A := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$A \xrightarrow{II + I} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} =: B$

B is not invertible, because $\forall C \in \mathbb{R}^{2\times 2}$: BC has zero rows

(b) $\Rightarrow$ A is not invertible.

Theorem 44: Let $A \in \mathbb{R}^{m\times m}$.

(a) Suppose $\exists\, B \in \mathbb{R}^{m\times m}: AB = I_m$

Then $BA = I_m$, i.e.
A is invertible and $A^{-1} = B$

(b) Suppose $\exists\, B \in \mathbb{R}^{m\times m}: BA = I_m$

Then $AB = I_m$.

Proof: (a) Let $R = EA$ be a reduced row echelon form of A.

If $R \neq I_m = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

then R has a zero row on the bottom $R = \begin{pmatrix} & & \\ 0 & \cdots & 0 \end{pmatrix}$.

$\Rightarrow E = EI_m = EAB = RB = \begin{pmatrix} & * & \\ 0 & - & 0 \end{pmatrix}$

$\lightning$ (contradiction) because E is invertible by Lemma 41 (c).

Thus $EA = I_m$ and $E = B$

(b) By (a) B is invertible and $B^{-1} = A$.

Thus A is invertible and $A^{-1} = B$. $\square$

<u>Example 4.5:</u>

(a) Invert $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\text{II}-\text{I}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-\text{II}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & \\ 0 & 1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{\text{I}-2\text{II}} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

Check: $A \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

Apply Theorem 44 (a).

(b) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Find an equivalent condition on a, b, c, d such that A is invertible and compute A^{-1} in that case.

End of Lecture 13.10.2023

Theorem 46: Let $A \in \mathbb{R}^{m \times m}$.

Then are equivalent:

1° A is invertible

2° The equation $Ax = 0$, $x \in \mathbb{R}^{m \times 1}$ has exactly one solution.

3° ~~The equation~~

Every row echelon form of A has no zero row.

Example 47:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ -2 & -7 & 5 \end{pmatrix}$$

Is A invertible?

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & & \\ 1 & 2 & -1 & & 1 & \\ -2 & -7 & 5 & & & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & & \\ 0 & 3 & -3 & -1 & 1 & \\ 0 & -9 & 9 & 2 & & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & & \\ 0 & 1 & -1 & -\frac{1}{3} & \frac{1}{3} & \\ \underbrace{0 \quad 0 \quad 0} & & & -1 & 3 & 1 \end{array}\right)$$

zero row $\Rightarrow$ A is not invertible.

Proof (Theorem 4.6):

$1° \Rightarrow 2°$: $Ax = b \Leftrightarrow x = A^{-1}b$

$$(A^{-1}(Ax) = A^{-1}b$$

$$(A^{-1}A)x = I_m x = x)$$

Thus $A^{-1}b \in \mathbb{R}^{m\times 1}$ is the only solution of $Ax = b$.
Take $b = 0 \Rightarrow 2°$.

$2° \Rightarrow 3°$: Assume $EA = B$

~~is~~ a row echelon form for A with a zero row.

$$B = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & 1 \\ 0 & \cdots & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}$$

Then $\exists j \in \{1, \dots, m\}$: The jth column does not contain a leading 1.

$\Rightarrow$ ~~Col~~ column operation $C \in \mathbb{R}^{n\times n}$ such that

$$BC = \begin{pmatrix} & \vdots & \\ & j & \\ & 0 & \\ & \vdots & \\ & 1 & \\ & \vdots & \\ & 0 & \end{pmatrix} =: \tilde{A}$$

Substitute $\tilde{x} = C^{-1}x$.

By 2° $0 = \tilde{A}\tilde{x}$ has only one solution.

↯ because it has

$\tilde{x} = 0$ and $\tilde{x} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} - j$

as solutions.

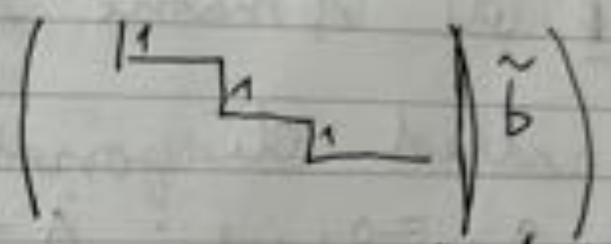
3° $\Rightarrow$ 1° Exercise $\square$

<u>Exercise 48:</u> Prove Theorem 25.

Definition 49: (notions for linear systems)

1) variables:

$Ax = b$ a linear system.

$(A|b) \longrightarrow$ 

If $i_1, \dots, i_l$ are the pivot columns then we call

$x_{i_1}, \dots, x_{i_l}$ pivotal variables

x_j, $j \neq i_1, \dots, i_l$ free variables.

Example 50:

Say $(A|b) \longrightarrow \left(\begin{array}{cccccc|c} 0 & 1 & 2 & 0 & 3 & 0 & 2 \\ & & & 1 & 2 & 0 & 1 \\ & & & & & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

parametric form of sol^n:
- $x_1 = s$, $x_2 = 2 - 2t - 3u$
- $x_3 = t$, $x_4 = 1 - 2u$
- $x_5 = \cancel{\ }\, u$, $x_6 = -1$

pivotal variables x_2, x_4, x_6

free — " — x_1, x_3, x_5.

Def 51: (a) A matrix $A = (a_{ij}) \in \mathbb{R}^{m\times m}$ is called diagonal if $\forall i \neq j \; a_{ij} = 0$, i.e. $A = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{mm} \end{pmatrix}$

(b) A matrix $A \in \mathbb{R}^{m\times m}$ is called upper ~~diagonal~~ ~~diagonal~~ triangular if

$\forall j > i : \; a_{ij} = 0$, i.e.

$$A = \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{mm} \end{pmatrix}$$

(c) Exercise: Define lower ~~diagonal~~ triangular matrix.

(d) A matrix $A \in \mathbb{R}^{m \times m}$ is called symmetric, if $a_{ij} = a_{ji} \quad \forall i,j$, i.e.

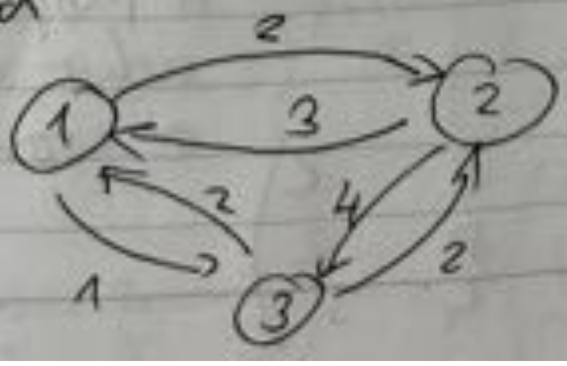
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{12} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{1m} & & & a_{mm} \end{pmatrix}$$

<u>Ex</u>: $\begin{pmatrix} 1 & 2 & 5 \\ 2 & ? & 3 \\ 5 & 3 & 1 \end{pmatrix}$

<u>Example 52</u>: (network, ~~directed~~ oriented graph)

Three cities C_1, C_2, C_3.
inhabitants in tausend: $10, 15, 7$

work traffic diagramm



in tausend

$A = (a_{ij})$ a_{ij} = number of inhabitants of C_i going to work in C_j

for $i \neq j$. (in tousend)

a_{ii} = # people in C_i who do not work outside C_i. (in tousend)

$$A = \begin{pmatrix} 7 & 2 & 1 \\ 3 & 8 & 4 \\ 2 & 2 & 3 \end{pmatrix}$$ incidence matrix

$B = (b_{ij})$ b_{ij} = # people coming to work in C_i living in C_j"
$i \neq j$

b_{ii} = # people of C_i who don't work or work in C_i.

$$B = \begin{pmatrix} 7 & 3 & 2 \\ 2 & 8 & 2 \\ 1 & 4 & 3 \end{pmatrix}$$

Def 53: Let $A \in \mathbb{R}^{m \times n}$, $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

The matrix

$$A^T := (a_{ij})_{\substack{1 \le j \le n \\ 1 \le i \le m}}$$

$$= \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & a_{m2} \\ a_{13} & a_{23} & \ddots & a_{m3} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

is called the transpose of A

Example 54: (a) in Example 52

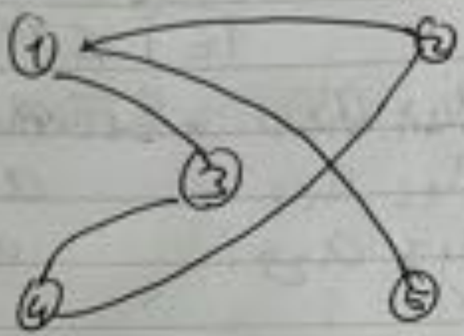
$$B = A^T.$$

(b) $(1, 2, -3)^T = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & 7 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ -3 & 7 \end{pmatrix}$

(d) $A \in \mathbb{R}^{m \times m}$ is symmetric iff
$A^T = A$.

(e) 5 persons P_1, P_2, P_3, P_4, P_5



edge means they are friends

incidence matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$A = A^T$ so A is symmetric

1.4 Application

Leontief Input-Output models

(Wassily Leontief)

To analyse economy:

Step 1: Divide the economy into sectors.

for example:

manufacturing (M)
agriculture (A)
utility (energy) (U)

consumption (C)

Step 2: determine relations between those sectors. (input-output)

O is called an open sector
because O has no output.

M needs input from A and
U to produce output
similar for
A and U.

Consumption matrix: C:

		M	A	U	
provider	M	0,5	0,1	0,1	in $\bar{\pi}$
	A	0,2	0,5	0,3	
	U	0,1	0,3	0,4	

input needed to produce
1 $\bar{\pi}$ output

M need input: 0,5 $\bar{\pi}$ from M
0,2 $\bar{\pi}$ from A
0,1 $\bar{\pi}$ from U
to produce 1 $\bar{\pi}$ output.

The open sector O has the following demand:

$$d_M = 7900 \text{万元}$$

$$d_A = 3950 \text{万元}$$

$$d_U = 1975 \text{万元}$$

$$d = \begin{pmatrix} d_M \\ d_A \\ d_U \end{pmatrix}$$ "outside demand vector"

Let $\begin{pmatrix} x_M \\ x_A \\ x_U \end{pmatrix}$ be the vector

telling the produced amount in 万元.

Then

$$\underbrace{x}_{\text{produced amount}} - \underbrace{Cx}_{\text{intermediate demand}} = \underbrace{d}_{\text{outside demand}}$$

Compute x. :

$$(I - C)x = d$$

If $(I-C)$ is invertible then

$$x = (I-C)^{-1}d$$

<u>Def 55</u>: An economy where $(I-C)$ is invertible and $(I-C)^{-1}$ has only non-negative entries is called <u>productive</u>

<u>Compute x</u>:

$$\left(\begin{array}{c|c} I - C & \begin{matrix} 7900 \\ 3950 \\ 1975 \end{matrix} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & & & 27{,}500 \\ & 1 & & 33750 \\ & & 1 & 24750 \end{array}\right)$$

$$\underbrace{\begin{matrix} 27{,}500 \\ 33750 \\ 24750 \end{matrix}}_{x}$$

<u>Question 56:</u> Is in our example the economy productive?

<u>Theorem 57:</u> Let C be a consuption matrix of an economy. Suppose that all row sums are smaller 1, i.e.

$$\forall_{i=1,\dots,m}: \sum_{j=1}^{m} c_{ij} < 1$$

Then the economy is productive.

End of Lecture 18.10.2023

The proof of Thm. 57 is not part of the lecture, but will be provided here for further reading.

<u>Lemma 58:</u> Suppose $A \in \mathbb{R}^{m\times m}$ satisfies

for all $i \in \{1,\dots,m\}$: $|a_{ii}| > \sum_{\substack{j=1\\ j\neq i}}^{m} |a_{ij}|$

Then A is invertible.

Example 59:
$$\begin{pmatrix} 2 & -1 & 0{,}5 \\ 1 & -3 & 1 \\ 0{,}25 & -1 & -2 \end{pmatrix}$$

is invertible. $2 > 1 + 0{,}5$
$3 > 1 + 1$
$2 > 1 + 0{,}25$

Proof (of Lemma 58):

By Theorem 46 it is enough to solve
$$Ax = 0 \in \mathbb{R}^{m\times 1}, \ x \in \mathbb{R}^{m\times 1}$$

We have to show that there is no other solution than 0.

Assume $\exists\ x \neq 0$: x is a solution.

Let i_0 be an index s.t.
$$|x_{i_0}| \geq |x_i| \ \forall i \in \{1, \dots, m\}$$

Then $0 = \sum_{j=1}^{m} a_{i_0 j} x_j$

$$\Rightarrow \quad 0 = \left| \sum_{j=1}^{m} a_{i_0 j} x_j \right|$$

$$\geq |a_{i_0 i_0}| |x_{i_0}| - \sum_{j \neq i_0} |a_{i_0 j}| |x_j|$$

$$\geq \underbrace{|x_{i_0}|}_{>0} \underbrace{\left(|a_{i_0 i_0}| - \sum_{j \neq i_0} |a_{i_0 j}| \right)}_{>0}$$

because $x \neq 0$ and $|x_{i_0}|$ is maximal.

$$> 0 \qquad \lightning \qquad \square$$

Proof (of Theorem 57)

- $I - C$ is invertible by Lemma 58.
- C is a consumption matrix

$\Rightarrow$ All entries of $C, C^2, C^3, \ldots$ are non-negative.

- <u>Claim:</u>

Let $c_0 := \max_{i} \sum_{j=1}^{k} c_{ij} < 1$

Then for all $n \in \mathbb{N}$ we have $1 \leq i, j \leq m$

~~(c_{ij}^{n})~~ for $C^n = (c_{ij}^{(n)})$ (nth power of C) that $0 \leq c_{ij}^{(n)} \leq c_0^n$.

⌜<u>Pf:</u> <u>$n=1$:</u> by definition of c_0.

<u>$n>1$:</u>
$$c_{ij}^{(n)} = \sum_{k=1}^{m} c_{ik} c_{kj}^{(n-1)}$$
$$\underset{\text{(IH)}}{\leq} \sum_{k=1}^{m} c_{ik} c_0^{n-1}$$
$$\underset{\left(\sum_{k=1}^{m} c_{ik} \leq c_0\right)}{\leq} c_0^{n-1} \cdot c_0 = c_0^n$$
□⌟

- Then $D = \sum_{n=0}^{\infty} C^n$ is well-defined because $\sum_{n=0}^{\infty} c_{ij}^{(n)}$ ~~cong~~ converges for every i,j.

- $(I_m - C) D = \lim_{n\to\infty} (I_m - C) \sum_{k=0}^{n} C^k$

$$= \lim_{n\to\infty} (I_m - C^{n+1})$$

$$= I_m \text{ because}$$

$$c_{ij}^{(n+1)} \longrightarrow 0 \text{ for } n+1 \longrightarrow \infty$$

So D is the inverse of $I_m - C$, and by definition all entries of D are non-negative $\square$