Def 6.9: Let $1 \leq k \leq n$.

$G(k,n) := \{W \subseteq \mathbb{R}^n \mid W$ is a k-dimensional real subvector space in $\mathbb{R}^n\}$
is called the Grassmanian of k-planes in $\mathbb{R}^n$.

Note: $G(1,n) = \mathbb{R}P^{n-1}$

Topology on $G(k,n)$:

$$F(k,n) = \{A \in \mathbb{R}^{n \times k} \mid rk(A) = k\}$$

$$A \sim B \Leftrightarrow_{def} \exists g \in GL_k(\mathbb{R}) : A = Bg$$

Then $F(k,n)/\sim \; \simeq G(k,n)$

(bijection).

We use the quotient topology on $F(k,n)/\sim$ to define a topology on $G(k,n)$.

<u>Exercise:</u> (a) $G(k,n)$ is Hausdorff
(b) $G(k,n)$ is second countable

(c) $G(k,n)$ is compact

(d) $G(k,n)$ has the structure of a smooth manifold.

We give an example for (d):
$k=2$, $n=4$.
For $1 \le i < j \le 4$ we put
$V_{ij} := \{ A \in F(2,4) \subseteq \mathbb{R}^{4\times 2} \mid A_{ij} \text{ has rk } 2 \}$

Here $A_{ij} := \begin{pmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{pmatrix}$ if $A = (a_{st})_{\substack{1\le s\le 4 \\ 1\le t\le 2}}$

Then V_{ij} is open in $\mathbb{R}^{4\times 2}$ and therefore in $F(2,4)$, because it is given by $\det(A_{ij}) \neq 0$.

$U_{ij} := V_{ij}/\sim$ is open in

$F(2,4)/\sim$, because the projection is

an open map.

<u>Example</u>: $U_{1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix}$

$U_{2,3} = \begin{bmatrix} * & * \\ 1 & 0 \\ 0 & 1 \\ * & * \end{bmatrix}$

To get the chart consider

$\phi_{ij} : U_{ij} \longrightarrow \mathbb{R}^{2\times 2} \simeq \mathbb{R}^4$

$\phi_{ij}([A]) := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

if $[A]_\sim = \begin{bmatrix} b_{11} & b_{12} \\ 1 & 0 \\ 0 & 1 \\ b_{21} & b_{22} \end{bmatrix} \begin{matrix} \\ \leftarrow i \\ \leftarrow j \\ \\ \end{matrix}$

More precise:

$\phi_{12}([A]) = A_{34} A_{12}^{-1}$.

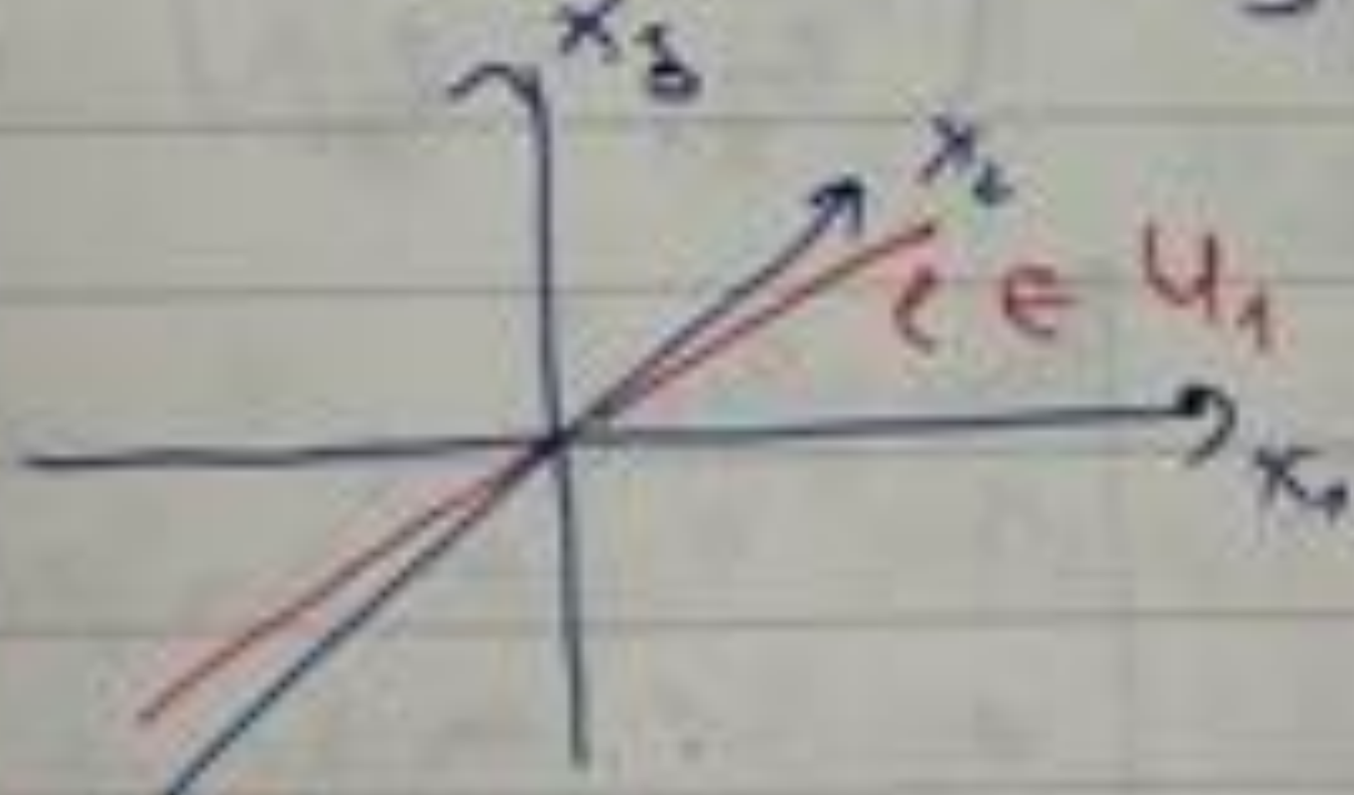
2.9.3

$U_{1,2} \triangleq \{ W \subseteq \mathbb{R}^4 \mid W$ 2-dim real v.s. such that $W \cap \mathrm{span}_{\mathbb{R}}\{e_3, e_4\} = \{0\} \}$

This is analogue to $\mathbb{RP}^2$:

$$U_1 = \{ [x_1 : x_2 : x_3] \mid x_1 \neq 0 \}$$
$$= \{ [1 : x_2 : x_3] \mid x_2, x_3 \in \mathbb{R} \}$$

= set of lines which intersect the $x_2 - x_3$ -plane in zero.



<u>Claim:</u> (a) $\phi_{1,2}$ is a homeomorphism

(b) $\phi_{12} \circ \phi_{23}^{-1}$ is a diffeomorphism

Pf: (a)

$$\begin{array}{ccc} V_{12} & \xrightarrow{\tilde{\phi}_{12}} & \mathbb{R}^{2\times 2} \\ & \searrow^{\pi} \quad \nearrow_{\phi_{12}} & \\ & U_{12} & \end{array}$$

(a1) Let $C \subseteq \mathbb{R}^{2\times 2}$ be open

$\Rightarrow \pi^{-1}\phi_{12}^{-1}(C) = \tilde{\phi}_{12}^{-1}(C)$ is open

because $\tilde{\phi}_{12}$ is continuous.

$\Rightarrow$ $\phi_{12}^{-1}(C)$ is open.
↑
quotient topology

(a2) Take $D \subseteq U_{12}$ open

Then $\phi_{12}(D) = \tilde{\phi}_{12}(\pi^{-1}(D))$

$$= \tilde{\phi}_{12}\Big(\underbrace{\begin{pmatrix} 1 & 1 \\ * & * \\ * & * \end{pmatrix} \cap \pi^{-1}(D)}_{\{(1\ 1)\}\times H,\ H \subseteq \mathbb{R}^{2\times 2} \text{ open}}\Big)$$

$\Rightarrow$ H is open.

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$B \in \mathbb{R}^{2\times 2}$

$$\phi_{23}\,\phi_{12}^{-1}(B) = \phi_{34}\left(\begin{bmatrix} 1 \\ B \end{bmatrix}\right)$$

$$= \begin{pmatrix} 1 & 0 \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ b_{11} & b_{12} \end{pmatrix}^{-1}$$

for $B \in \phi_{12}(U_{12} \cap U_{23})$

$$= \left\{ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \middle| \; c_{ij} \in \mathbb{R},\ c_{12} \neq 0 \right\}$$

Exercise:
Give a geometric interpretation

Def 6.10: Let $1 \leq k \leq n$. The following vector bundle is called the Grassmann bundle over $G(k,n)$

$$\gamma_{k,n} := (p, E_{k,n}, G(k,n), \Phi)$$

$$E_{k,n} := \{ (P, v) \mid P \in G(k,n) \text{ and } v \in P. \}$$

$$p : E_{k,n} \longrightarrow G_{k,n} \quad (P, v) \longmapsto P.$$

The topology on $E_{k,n}$ is given as follows.

Take $W \in G(k,n)$, $W' \in G(n-k,n)$ s.t. $W \oplus W' = \mathbb{R}^n$. Set

- $E_{W'} := \{(P,v) \mid P \in G(k,n),\ v \in P$ and $P \oplus W' = \mathbb{R}^n\}$

$$= \{(P,v) \in E_{k,n} \mid W' \cap P = \{0\}\}$$

- $\pi^{W' \oplus W}_{W} : \mathbb{R}^n \longrightarrow W$

$$w + w' \longmapsto w$$

- $U_{W'} := \{P \in G(k,n) \mid P \cap W' = \{0\}\}$
- $\varphi_{W',W} : E_{W'} \longrightarrow U_{W'} \times W$

$$\varphi_{W',W}(P,v) := \pi^{W' \oplus W}_{W}(v)$$

We take on $E_{k,n}$ the coarsest topology such that

- all $E_{W'}$ are open, $W' \in G(n-k,n)$
- all $\varphi_{W',W}$ are continuous. $(W',W) \in G(n-k,n) \times G(k,n)$.

Date

<u>Claim:</u> The topology is equal to $\mathcal{I}_{k,n} := \{ F \subseteq E_{k,n} \mid$
$\forall W' \in G(n-k,n) \, \forall W \in G(k,n):$
$\varphi_{W',W}(E_{W'} \cap F)$ is open in $U_{W'} \times W \}$

<u>Proof:</u> ① $\mathcal{I}_{k,n}$ is a topology, because

a) $\emptyset, E_{n,k} \in \mathcal{I}_{k,n}$

b) Let $F_1, \dots, F_e \in \mathcal{I}_{k,n}$. Then $\forall_{W',W}$: $\varphi_{W',W}(E_{W'} \cap (F_1 \cap \dots \cap F_e))$
$= \varphi_{W',W}(E_{W'} \cap F_1) \cap \dots \cap \varphi_{W',W}(E_{W'} \cap F_e)$
is open in $U_{W'} \times W$.

c) For $F_i \in \mathcal{I}_{k,n}$, $i \in I$, we have $\bigcup_{i \in I} F_i \in \mathcal{I}_{k,n}$.

② $E_{W'} \in \mathcal{I}_{k,n}$, because

$\forall_{V',V}$: $\varphi_{V',V}(E_{V'} \cap E_{W'}) = (U_{V'} \cap U_{W'}) \times V$

is open in $U_{V'} \times V$.

③ Take $(W', W) \in Gr(n-k, n) \times G(k,n)$.
We have to show that $\varphi_{W',W}$ is continuous w.r.t. $\mathcal{I}_{k,n}|_{E_{W'}}$

③.1 $\forall_{V',V}$: $\varphi_{V',V} \circ \varphi^{-1}_{W',W} : (U_{W'} \cap U_{V'}) \times W$

$$\downarrow$$

$$(U_{W'} \cap U_{V'}) \times V$$

is a diffeomorphism.

<u>Proof:</u> Take $(V', V) \in Gr(k,n)$ and

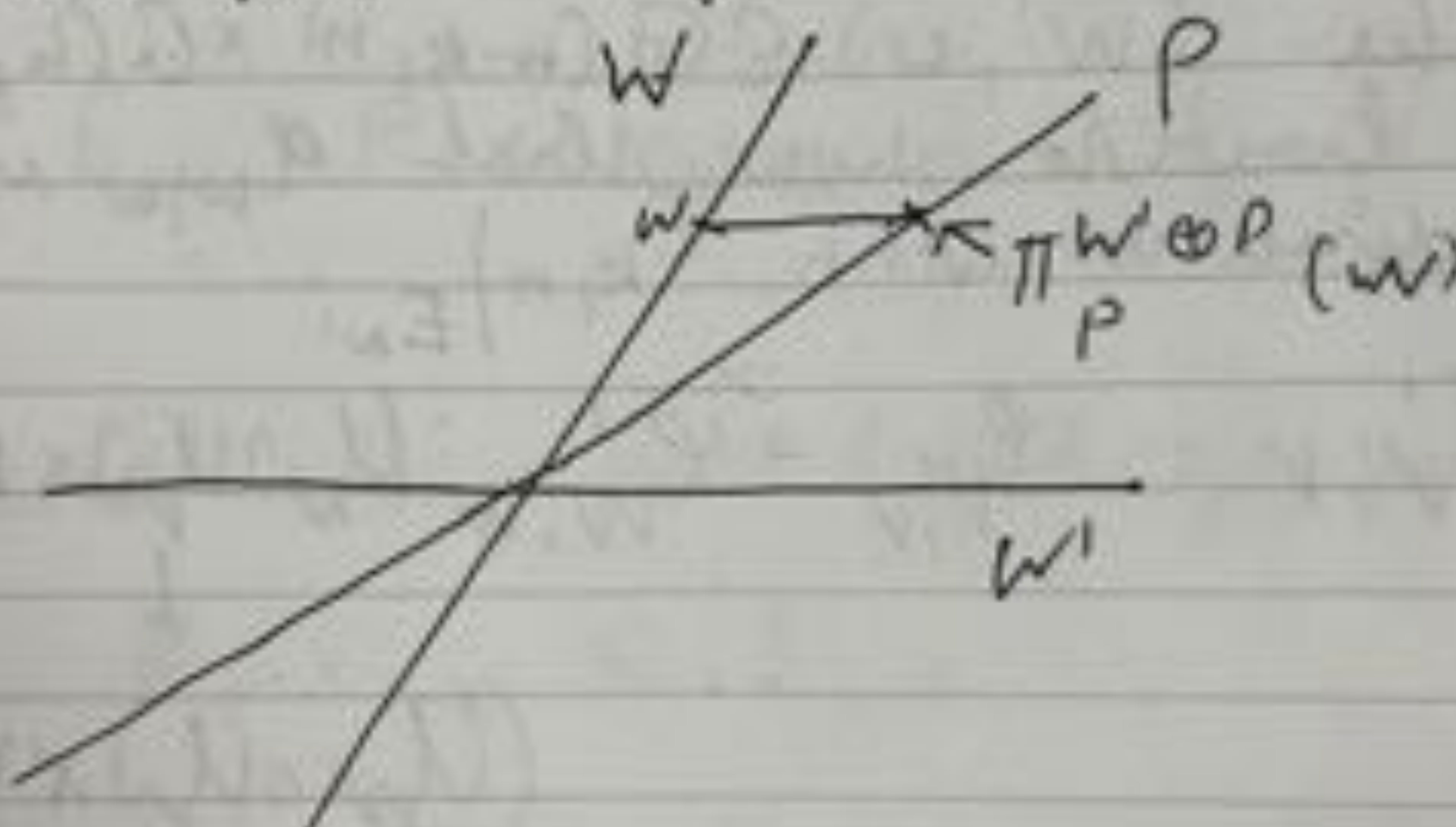
$(P, w) \in U_{W'} \times W$.

Then $\varphi^{-1}_{W',W}(P,w) = (P, \Pi_P^{W' \oplus P}(w))$

and $\varphi_{V',V}(P, \Pi_P^{W' \oplus P}(w)) = (P, \Pi_V^{V' \oplus V}(\Pi_P^{W' \oplus P}(w)))$

So $\varphi_{V',V} \circ \varphi_{W',W}^{-1}$ is C^∞.
(infact C^ω.)

Picture for $\pi_P^{W'\oplus P}$:



<u>Exercise</u>: $\pi_P^{W'\oplus P}$ is C^∞.

(Hint: Use a basis of W and W' and its dual and matrices as in Def. 6.9.)

(3.2) Take $Z \subseteq U_{W'} \times W$ open and $(V',V) \in G(n-k,n) \times G(k,n)$
Then

$\varphi_{V',V}\left(\varphi_{W',W}^{-1}(Z) \cap E_{V'}\right)$

$= \varphi_{V',V} \circ \varphi^{-1}_{W',W}\left(\tau \cap ((U_{W'} \cap U_{V'}) \times W)\right)$

is open because of (3.1). □

Claim: $\{ (\varphi_{W',W}, U_{W'}) \mid W' \in G(n-k,n) \}$

is a C^∞- vector bundle atlas.

Proof. Step 1: $(\varphi_{W',W}, U_{W'})$ is a vector bundle chart:

$$\begin{array}{ccc} E_{W'} & \xrightarrow{\varphi_{W',W}} & U_{W'} \times W \\ & {\scriptstyle p} \searrow \quad \circlearrowleft \quad \swarrow {\scriptstyle \pi_1} & \\ & U_{W'} & \end{array}$$

(~~[illegible]~~ $\varphi_{W',W}$ preserves fibers)

and $\varphi_{W',W}$ is a homeomorphism,

because $\varphi_{W',W}$ is continuous and open w.r.t. $\mathcal{I}_{k,n}$.

Step 2: C^∞- v.b. atlas: For C^∞, see (3.1).

For linearity on fibers:

note that $v \mapsto \Pi_V^{V' \oplus V}(\Pi_P^{W' \oplus P}(w))$
is a linear map if
we fix P.

(End of Def 6.10)

—— End of Lecture 23.05.2023

Let us sum up what we did to prove that an object is a smooth vector bundle.

<u>Theorem 6.11:</u> (vector bundle criterion

Let M be a mf. and $E \xrightarrow{p} M$ be a surjective map such that

(VC1) $\forall_{x \in M}$: $p^{-1}(x) =: E_x$ is a vector space of dim k.

(VC2) $\exists$ atlas $(\varphi_\alpha, U_\alpha)_{\alpha \in A}$ of M such th

(i) $\forall_{\alpha \in A} \exists \; \Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}$

such that Φ_α preserves fibers and is there a linear isomorp

(ii) $\forall_{\alpha, \beta}$ with $U_\alpha \cap U_\beta \neq \emptyset$:

$\Phi_\beta \circ \Phi_\alpha^{-1}$ has the form

$(x, v) \longmapsto (x, \tau_{\beta,\alpha}(x)v)$ for some smooth $\tau_{\beta,\alpha}: U_\alpha \cap U_\beta \to GL_n(\mathbb{R})$

Then there exists a topology on E and a C^∞ structure $\Phi_{(\infty)}$ on (p, E, M) such that $(p, E, M, \Phi_{(\infty)})$ is a smooth vector bundle on M.

Proof: $\mathcal{T} := \{ F \subseteq E \mid \forall_\alpha : \Phi_\alpha(F \cap p^{-1}(U_\alpha))$ is open in $U_\alpha \times \mathbb{R}^n \}$

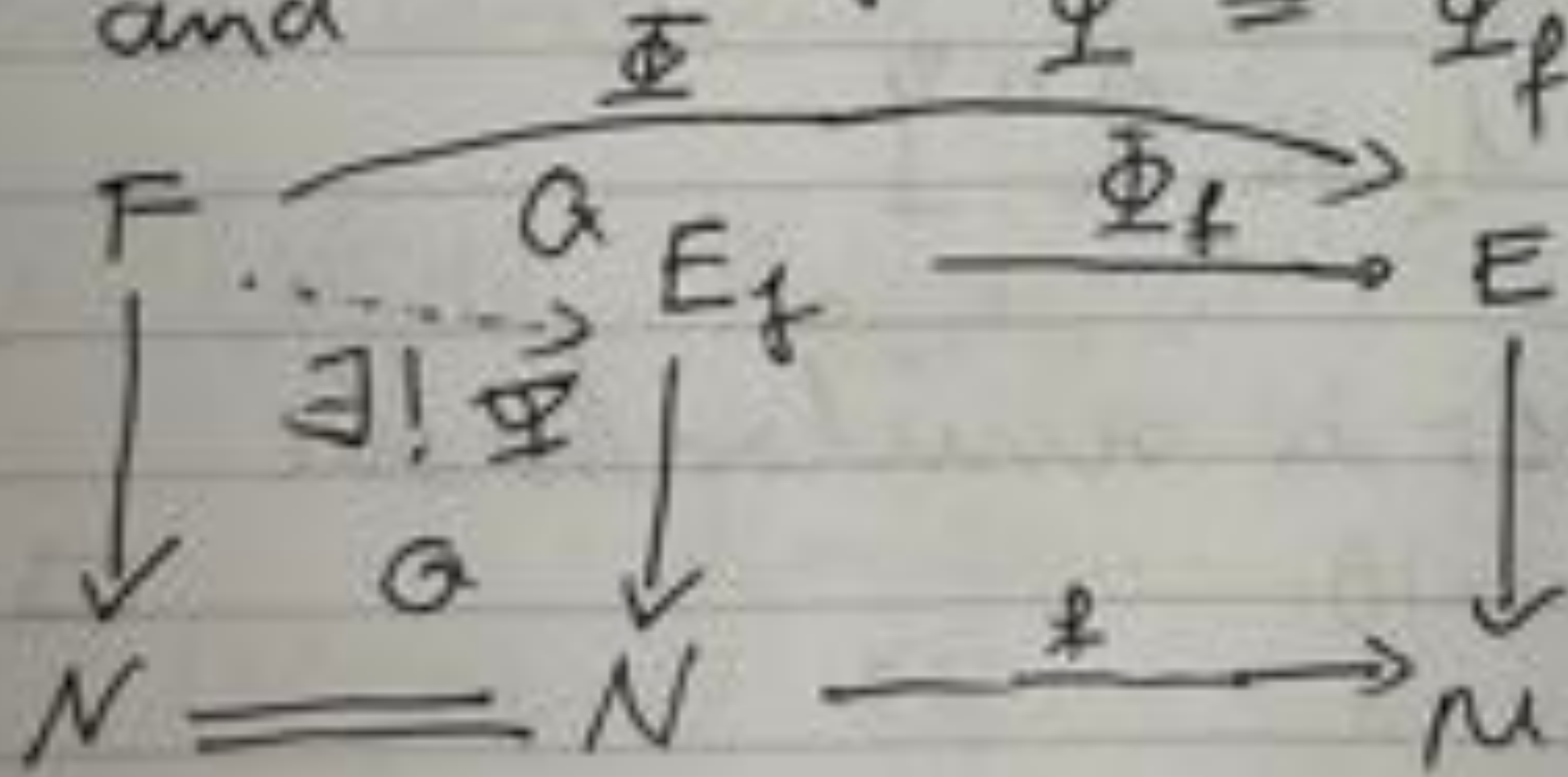
The proof now is similar to the proof in Def. 6.10. □

An application of Theorem 6.11 is the pull back of a vector bundle.

Prop 6.12: Let (p_M, E_M, M) be a vector bundle over a mf M.
Let $f \in C^\infty(N, M)$.
Then there exist a vector bundle (p_f, E_f, N) over N and a morphism of

$\Phi_f : E_f \to E_M$ such that for

all vb (q, F, N) and morphisms $\Phi : F \to E_M$ covering f
there exists a unique vb. morphisms $\Psi : F \to E_f$ such that Ψ covers
and $\Phi = \Phi_f \circ \Psi$.



Proof: $E_f = \{ (Q, v) \mid Q \in N, (f(Q), v) \in p_M^{-1}(f(Q)) \}$

(VC1) $E_{f,Q} = E_{f(Q)}$ is linear.

(VC2) Take an atlas $(\varphi_\alpha, U_\alpha)$ for M satisfying (VC2) with Φ_α.
Take $V_\alpha := f^{-1}(U_\alpha)$ Then we define

(i) $$\Psi_\alpha : p_f^{-1}(V_\alpha) \longrightarrow V_\alpha \times \mathbb{R}^n$$

via $$\Psi_\alpha(Q, v) := (Q, \Phi_\alpha(f(Q), v)_2)$$

Then $$\Psi_\alpha(Q, v)_2 = \Phi_\alpha(f(Q), v)_2,$$

i.e. $\Psi_\alpha(Q, -)_2 = \Phi_\alpha(f(Q), -)_2$ is a linear isomorphism.

(ii) $$(\Psi_\beta \circ \Psi_\alpha^{-1}(Q, v))_2 = (\Phi_\beta \circ \Phi_\alpha^{-1}(f(Q), v))_2$$

Thus $$I_{\Psi, \beta, \alpha} = I_{\Phi, \beta, \alpha} \circ f$$
$$\in C^\infty(V_\alpha \cap V_\beta, GL_n(\mathbb{R}))$$
The universal property is an exercise □

<u>Example 6.13:</u> 1) $C := S^1 \times \mathbb{R} \subseteq \mathbb{R}^3$

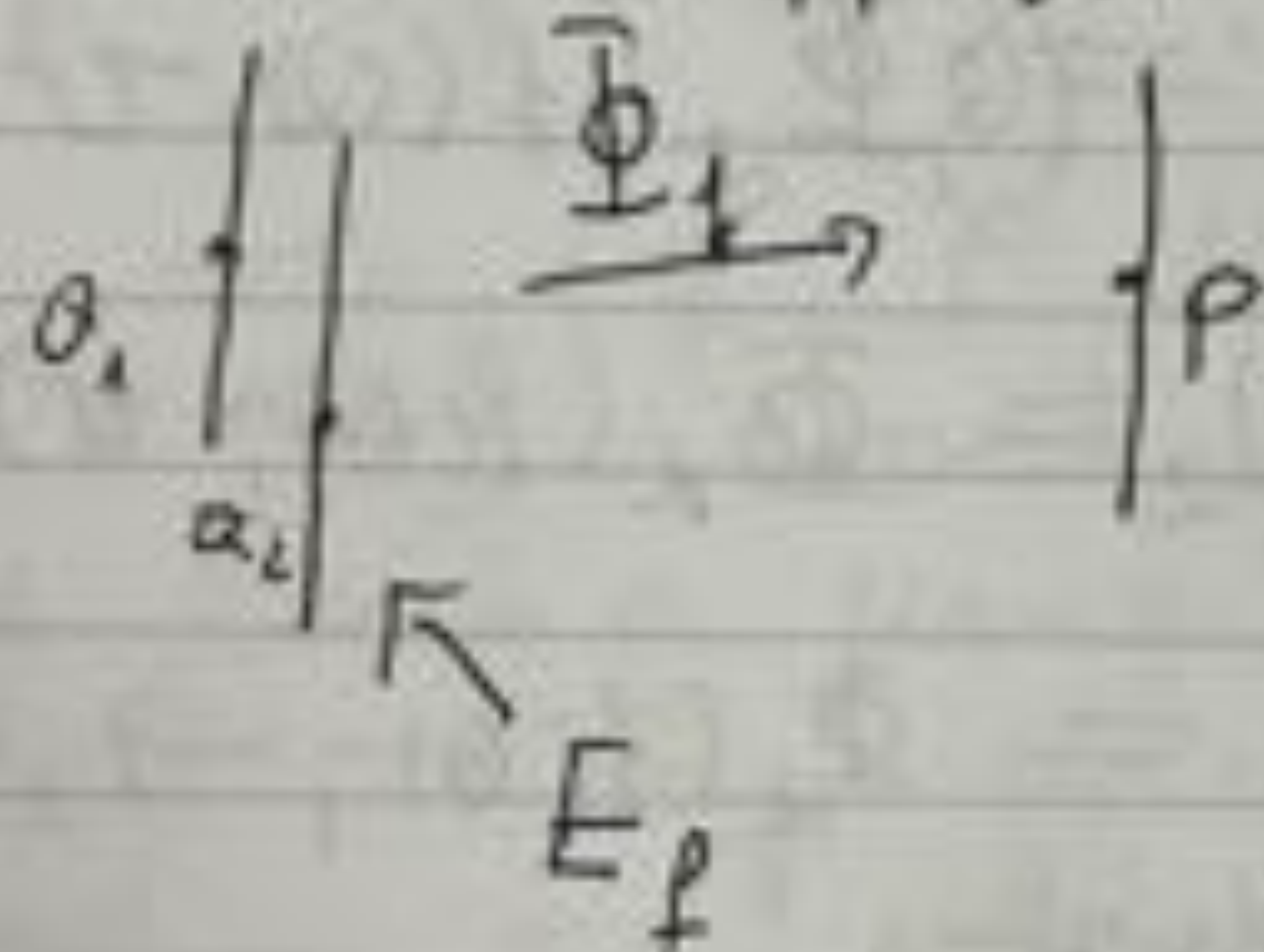
infinite cylinder.

$$f : C \longrightarrow S^1 \simeq \mathbb{R}P^1$$
$$(x,y,z) \longmapsto (x,y)$$

visualize $f^* \gamma_{1,2}$.

2) $M = \{pt\}$ $\quad E = \{pt\} \times \mathbb{R}$

$N = \{Q_1, Q_2\}$ $\quad f(Q_i) := P$.



We now want to sketch the proof of the classification theorem

__Def 6.14__ ~~let~~

let ξ be a k-vector bundle on a manifold M. let $1 \leq k \leq n$. A map ~~Suppose~~ $f \in C^\infty(M, G(k,n))$ is called __classifying map__ for ξ if $f^* \gamma_{k,n} \cong \xi$.

Theorem 6.15. (Classification theorem, see II 43.4)

Let ξ be a k-plane bundle on an m-dim. manifold M. Let $n \in \mathbb{N}$. Suppose $n \geq k+m$.
Then there exists a classifying map for ξ in $C^\infty(M, G(k,n))$.

The proof uses several concepts:
- Baire property
- transversality
- globalization theorem.

Examples 6.16:

(a) $\xi = \varepsilon^k_M$ Consider $M \xrightarrow{f} G(k,k) = \{pt\}$
$p \longmapsto pt.$

Then $\varepsilon^k_M = f^* \gamma_{kk}$

(γ_{kk} is the trivial k-plane bundle over a point)

(b) $M = S^2$ Consider the two bundles on S^2:

ξ_1 = normal bundle on S^2

$\xi_2 = TS^2$.

ξ_1 is trivial: $E_1 = \{(p, \lambda p) \mid p \in S^2, \lambda \in \mathbb{R}\}$

$$(p, \lambda p) \mapsto (p, \lambda), \quad E_1 \cong S^2 \times \mathbb{R}$$

ξ_2 is non-trivial, because of the hairy ball theorem.

(**Thm** (hairy ball theorem): Let X be a smooth vector field on S^2. Then X has a zero.)

$\xi_2 = f^* \gamma_{2,3}$ where $S^2 \xrightarrow{f} \mathbb{RP}^2 \cong G(2,3)$

is identifying ~~antipodal~~ anti-podal points.

$f((x,y,z)) := [x : y : z]$

(c) If M is compact we get an easy proof of the theorem for n big enough.

Take a finite open covering M, say $(U_i)_{i\in I}$ such that on every U_i the restriction $\xi|_{U_i}$ is trivial, i.e. $\xi|_{U_i} \overset{\Phi_i}{\cong} U_i \times \mathbb{R}^k$.

Let $(\lambda_i)_{i\in I}$ be a partition of unity subordinate to $(U_i)_{i\in I}$

<u>Claim</u>: Let $\delta := |I|$ then there exists a monomorphism of v.b. $\xi \xrightarrow{F} M \times \mathbb{R}^{\delta k}$ covering id_M.

<u>Proof</u>: $I = \{1, \dots, \delta\}$. We define

$$F(v) := (p, \lambda_1(p)\Phi_1(v)_2, \lambda_2(p)\Phi_2(v)_2, \dots \lambda_\delta(p)\Phi_\delta(v)_2), \; v \in E_p.$$

Then F is a monomorphism of v.b. □

Now. ~~At~~ take a monomorphism $F: \xi \longrightarrow M \times \mathbb{R}^{\delta k}$ covering id_M.

Define $f: M \longrightarrow G(k, \delta k)$

via $f(p) := F_2(E_p)$

<u>Exercise</u>: f is smooth.

<u>Claim</u>: $\xi \simeq f^* \gamma_{k,n}$, $n := \delta k$

<u>Proof</u>: $p \in M$, $v \in E_p$ $H(v) := (p, F(v)_2$

H_p is a linear isomophism $E_p \longrightarrow E_{f(p)} = F_p(E_p)$

because $H_p = F_p$ is injective.

To show: H is smooth
<u>Proof</u>: $H: E \longrightarrow E_{k,n,f}$

Take $p_0 \in M$. and put $W := F(E_{p_0})$

Take $W' \in G(n-k, n)$ such that $W \oplus W' = \mathbb{R}^n$. Put $V_{W'} := f^{-1}(U_{W'})$

$$\varphi := \varphi_{W';W} : E_{W'} \longrightarrow U_{W'} \times W$$

is a v.b. chart for $\gamma_{k,n}$. By definition of the pullback of a v.b.

$$\Psi : (E_{k,n})_f\big|_{V_{W'}} \longrightarrow V_{W'} \times W$$

given by $\Psi(p, v) := (p, \varphi(f(p), v)_2)$

is a v.b. chart of $f^* \gamma_{k,n}$.
Then $\Psi \circ H : E_{V_{W'}} \longrightarrow V_{W'} \times W$

is smooth, because we have for $v \in E_{V_{W'}}$:

~~$\Psi \circ H(v) = (p(P), \varphi(f(p(P)), F(v)_2$~~

$$\Psi \circ H(v) = (p(v), \varphi(f(p(v)), F(v)_2)_2)$$

End of Lecture 26.05.2023 $\square$