To ensure the existence of regular values we have Sard's Theorem.

<u>Theorem (Sard) 5.40</u>: Let M^m, N^n, $m,n \geq 1$ be manifolds and $f \in C^\infty(M,N)$. Then the set of critical values of f is a set of measure zero in N.

<u>Recall</u>: (1) $L \subseteq N$ has measure 0 iff $\varphi(L \cap U)$ — " — for all charts (φ, U) of N.

(2) Let $\Sigma_f \subseteq M$ be the set of critical ~~points~~ points of f. Then $f(\Sigma_f)$ is the set of critical values.

<u>Example 5.41</u>: 1) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) = x^2$$

Then $\Sigma_f = 0 \times \mathbb{R}$

and $f(\Sigma_f) = 0$.

2) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$. $f(x,y) = 1$. $\forall x,y \in \mathbb{R}$

$\Sigma_f = \mathbb{R}^2$, $f(\Sigma_f) = \{1\}$.

Proof: We only need to prove the assertion locally so we can restrict to
$$M = U \subseteq \mathbb{R}^m \text{ and } N = \mathbb{R}^n.$$

Induction on m: $m=0$ ✓, because then $f(U)$ is countable in $\mathbb{R}^n$, so has measure zero.

$m>0$: Put for $k \geq 1$:
$$\Sigma_k := \{ \underline{x} \in U \mid \text{all partial derivatives up to order } k \text{ are } 0 \}$$

Then $\Sigma_f \supseteq \Sigma_1 \supseteq \Sigma_2 \supseteq \dots$.

The proof is done in 3 Steps:

Step 1: $f(\Sigma_f \setminus \Sigma_1)$ has measure 0.

Step 2: $f(\Sigma_k \setminus \Sigma_{k+1})$ has measure 0 for $k \geq 1$

Step 3: $f(\Sigma_k)$ has measure 0 for k big enough.

Proof of Step 1: $m=1$ ✓, $m>1$:

Take $p \in \Sigma_f \setminus \Sigma_1$ w.l.o.g. $\frac{\partial f_1}{\partial x_1}(p) \neq 0$.

Since we only need to prove the Step locally and by the Inverse Function Theorem we can assume w.l.o.g.

$$f(\underline{x}) = (x_1, f_2(\underline{x}), f_3(\underline{x}), \dots, f_n(\underline{x}))$$

(Just compose with the local inverse of $h(\underline{x}) = (f_1(\underline{x}), x_2, \dots, x_m)$.)

Then $f_t : (\{t\} \times \mathbb{R}^{m-1}) \cap U =: U_t \to \{t\} \times \mathbb{R}^{n-1}$

$$f_t(t, x_2, \dots, x_m) := f(t, x_2, \dots, x_m)$$

satisfies (i) $\Sigma_{f_t} = \Sigma_f \cap U_t$, because

$$D(f) = \begin{pmatrix} 1 & \\ * & D(f_t) \end{pmatrix}$$

(ii) $f_t(\Sigma_{f_t}) = f(\Sigma_f) \cap (\{t\} \times \mathbb{R}^{n-1})$

iii $f_t(\Sigma_{f_t})$ has measure zero by (IH).

Now apply Fubini to obtain from (ii) and (iii) that

$f(\Sigma_f \setminus \Sigma_1)$ has measure zero in $\mathbb{R}^n$

<u>Step 2:</u> Take $p \in \Sigma_k \setminus \Sigma_{k+1}$ s.t. w.l.o.g.

$$\frac{\partial^{k+1} f}{\partial x_1 \partial x_{i_2} \cdots \partial x_{i_{k+1}}}(p) \neq 0.$$

Put $h(\underline{x}) = (w(\underline{x}), x_2, \dots, x_m)$ with $w(\underline{x}) := \frac{\partial^k f}{\partial x_{i_2} \cdots \partial x_{i_{k+1}}}(\underline{x})$

and consider V an open nbhd of p s.t.

$h : V \longrightarrow V'$ is a diffeom.

$$g := f \circ h^{-1} : V' \longrightarrow \mathbb{R}^n$$

$$f(V \cap (\Sigma_k \setminus \Sigma_{k+1})) \subseteq g((0 \times \mathbb{R}^{m-1}) \cap V')$$

Then for $g_0 : (0 \times \mathbb{R}^{m-1}) \cap V' \longrightarrow \mathbb{R}^n$

$g_0(\Sigma_{g_0})$ has measure zero in $\mathbb{R}^n$.

Thus $f(V \cap (\Sigma_k \setminus \Sigma_{k+1}))$ has measure zero in $\mathbb{R}^n$.

<u>Step 3:</u> Let $k > \frac{m}{n} - 1$.

Take a cube I of edge length δ

Then for $\underline{x} \in I \cap \Sigma_k$ and $\underline{h}$ with $\underline{x} + \underline{h} \in I$ we have

$$|f(\underline{x} + \underline{h}) - f(\underline{x})|_2 \leq c \cdot |\underline{h}|_2^{k+1}$$

for $c \in \mathbb{R}$ only depending on f and I.

Subdivide I in cubes of edge length $\frac{\delta}{r}$.

For such a cube I_1 we have that $f(I_1)$ lies in a cube of edge length $d \cdot \frac{1}{r^{k+1}}$

if $I_1 \cap \Sigma_k \neq \emptyset$.

d does only depend on m, δ and c.

$f(I \cap \Sigma_k)$ is contained in the union of at most r^m such cubes.

$$\Rightarrow \lambda_n(f(I \cap \Sigma_k)) \leq r^m \left(\frac{d}{r^{k+1}}\right)^n$$

$$= d^n r^{m-(k+1)n} \xrightarrow[r \to \infty]{} 0.$$

$\square$

At last of the chapter let us define the Euler characteristic of a manifold

<u>Def 5.42</u>: Let M^m be a mf. such that all $H^k_{dR}(M)$ are finite

dimensional. The number
$\chi(M) := \sum_{k\geq 0} (-1)^k \dim_{\mathbb{R}} H^k_{dR}(M)$ is
called the Euler characteristic of M.

Example 5.43: 1) $\chi(\mathbb{R}^n) = 1$, $n \geq 0$

2) $\chi(L_g) = 1 - 2g + 1 = 2 - 2g$

3) $\chi(\text{Möb}) = \chi(S^1) = 1 - 1 = 0$

4) $\chi(\mathbb{R}P^2) = 1 - 0 + 0 = 1$

5) n odd, $n \geq 1$, $\chi(\mathbb{R}P^n) = 0$.

6) $\chi(K^2) \underset{\uparrow}{=} 1 - (0+0+1) + 0 = 0$

Prop. 5.23

$K^2 = \mathbb{R}P^2 \# \mathbb{R}P^2$

Exercise: Let M^m, N^m be closed connected mf, $m \geq 2$.
Compute $\chi(M \# N)$ in terms of $\chi(M)$ and $\chi(N)$.

Chapter VI

Vector bundles (Ref: Hirsch (II))

Def 6.1: Let $p : E \longrightarrow B$ be a continuous map.

(a) A vector bundle chart of (p, E, B) with domain U and dimension n

is a homeomorphism $\varphi : p^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^n$ where $U \subseteq B$ is open and

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

(i.e. φ maps $p^{-1}(x)$ to $\{x\} \times \mathbb{R}^n$.)

Given $x \in U$ we put $\varphi_x : p^{-1}(x) \longrightarrow \mathbb{R}^n$ to be $\varphi_x := \pi_2 \circ \varphi\big|_{p^{-1}(x)}$.

So for $y \in p^{-1}(x)$ we have
$$(x, \varphi_x(y)) = \varphi(y).$$

(with domains covering B)

(b) A family Φ of vector bundle charts on (p, E, B) is called vector bundle atlas on (p, E, B) if for $(\varphi, U), (\psi, V) \in \Phi$ and $x \in U \cap V$

$$\psi_x \varphi_x^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ is } \mathbb{R}\text{-linear.}$$

Then the map

$$U \cap V \longrightarrow GL_n(\mathbb{R})$$

$$x \longmapsto \psi_x \varphi_x^{-1} \quad \text{is}$$

continuous (Exercise). It is called the transition function of $(\varphi, U), (\psi, V)$.

(c) A maximal atlas Φ is called a <u>vector bundle structure on (p, E, B)</u>

Given a v.b. structure Φ on (p, E, B) we call $\xi = (p, E, B, \Phi)$ a vector bundle with
- (fibre) dimension n
- projection p
- total space E
- base space B.

For $A \subseteq B$ we write ξ_A for
$(p|_{p^{-1}(A)},\ p^{-1}(A),\ A,\ \Phi|_A)$

For example for $x \in B$ we get
for $A = \{x\}$:

$$\xi_x = \Big(p^{-1}(x) \to \{x\},\ \underbrace{p^{-1}(x)}_{= E_x},\ \{x\},\ \underbrace{p^{-1}(x) \xrightarrow[\varphi_x]{\sim} \mathbb{R}^n}_{\text{structure for}}\Big)$$

<u>Example 6.2:</u>

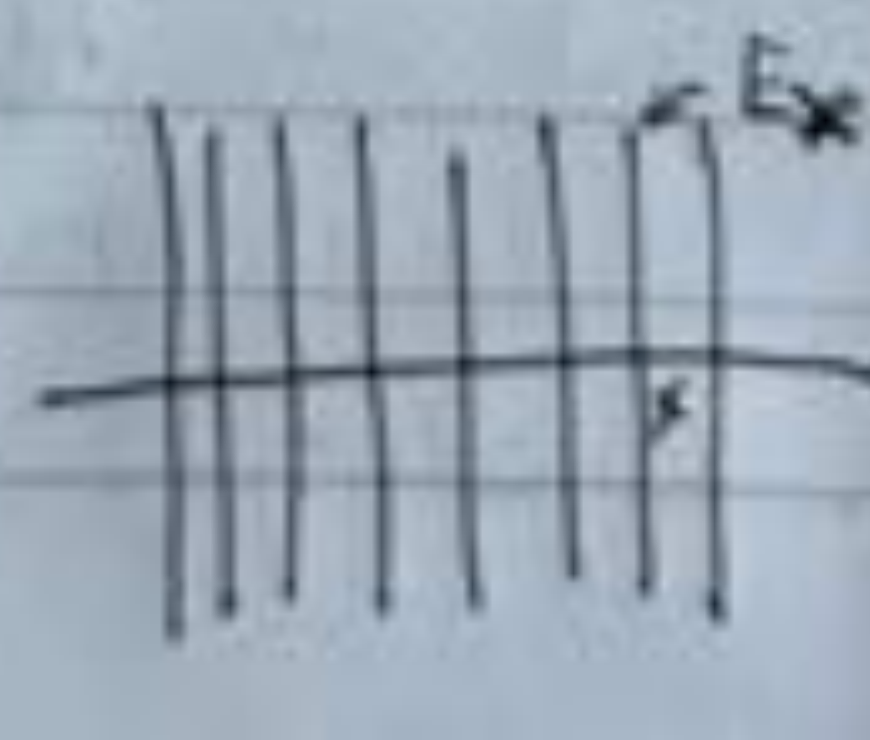
1) (trivial vector bundle of dimension n)
Let B be a top. space

$$p: B \times \mathbb{R}^n \longrightarrow B \quad (x, v) \longmapsto x$$

$(\varphi = \mathrm{id}_{B \times \mathbb{R}^n}, B)$ is a vector bundle chart on $(p, B \times \mathbb{R}^n, B)$

<u>Ex</u>: $B = D^2$

$B = \mathbb{R}$, $E = \mathbb{R}^2$
$p: E \to \mathbb{R}$ $(x,y) \mapsto x$



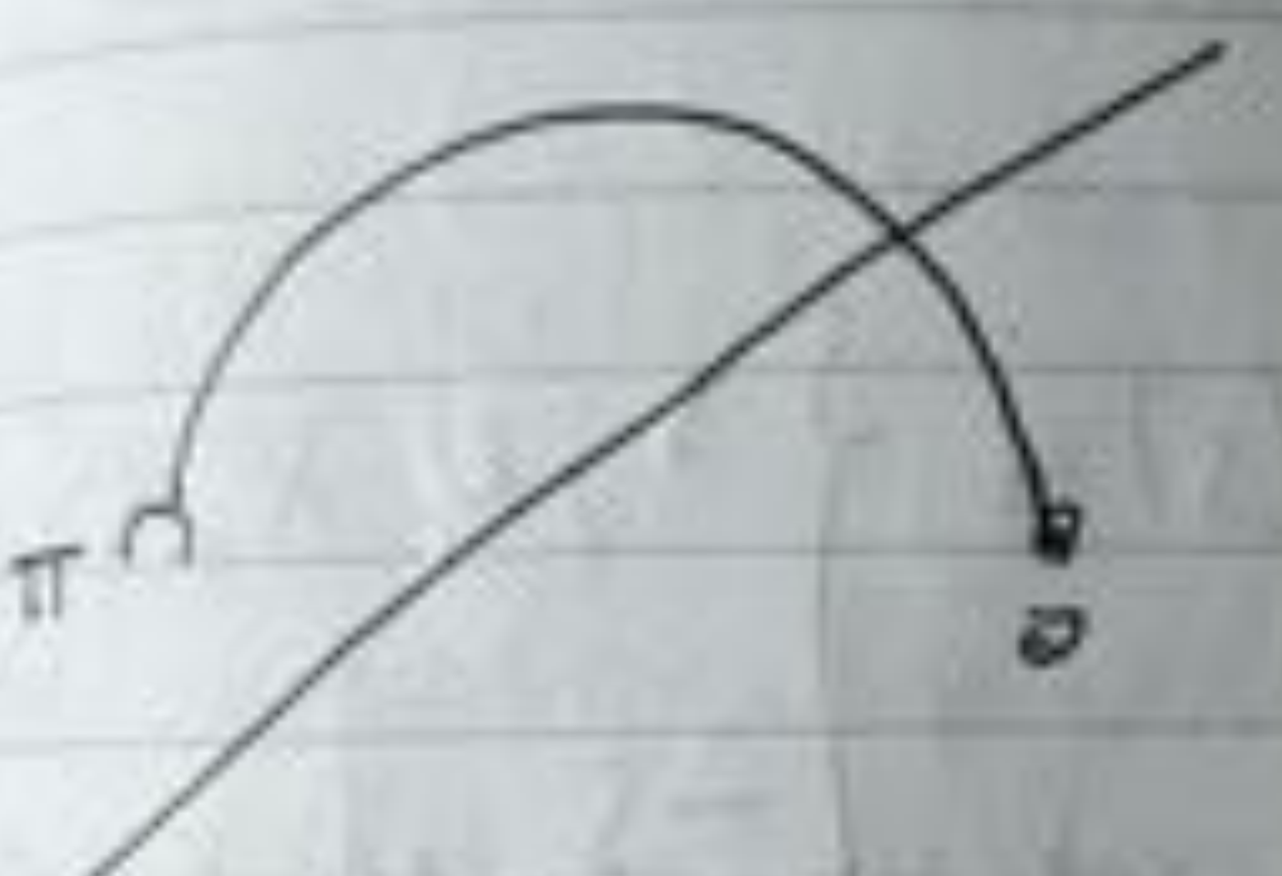
$\varepsilon_B^n := (p, B \times \mathbb{R}^n, B, \Phi)$ "trivial n-dim vb over B"

2) $B = \mathbb{R}P^1$, $n = 1$.

$$E = \left\{ ([x_0 : x_1], v) \mid \begin{array}{l} v \in \mathbb{R}(x_0, x_1) \\ (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0,0)\} \end{array} \right\}$$

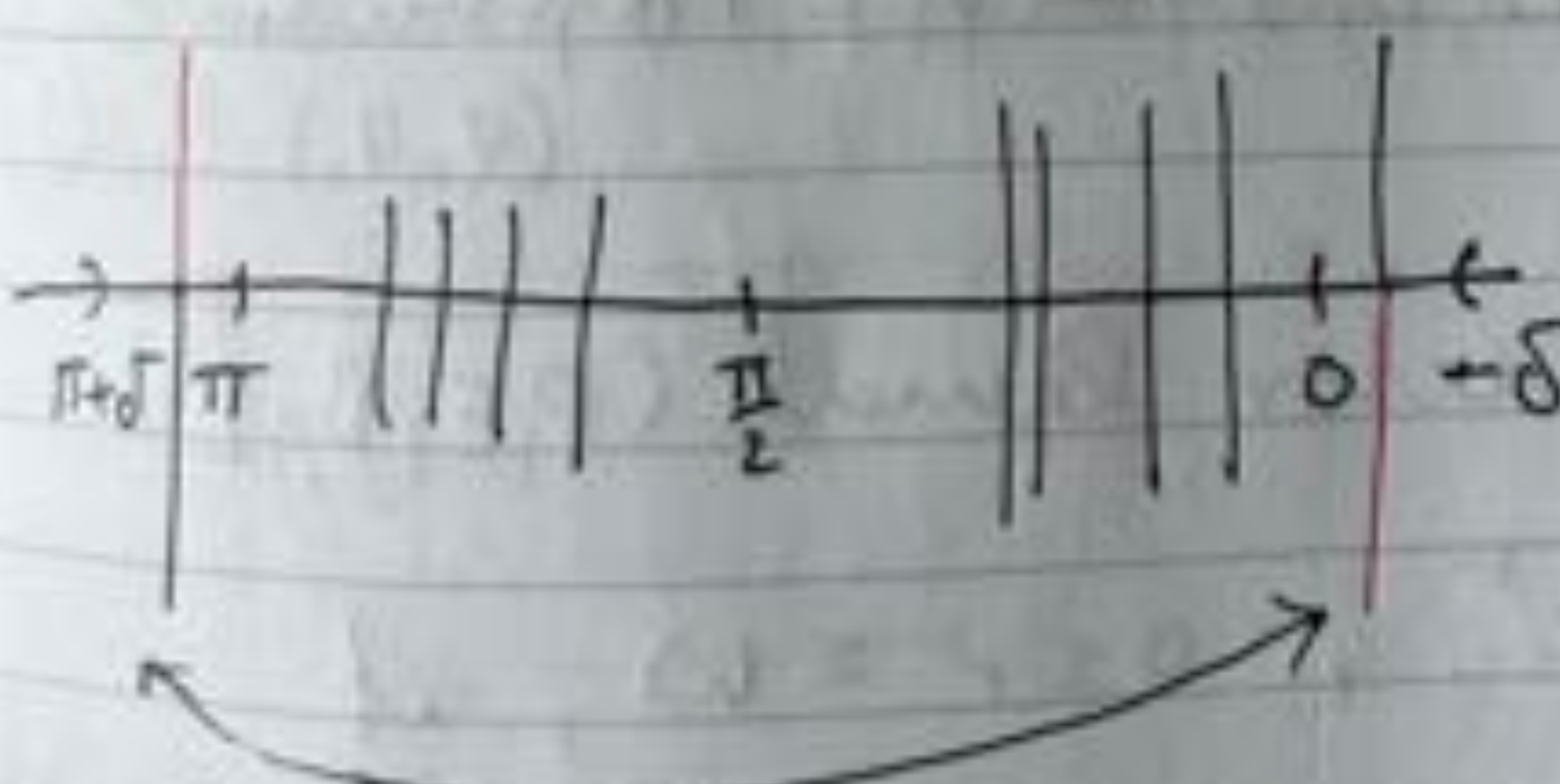
$p : E \longrightarrow B \quad p([x_0 : x_1], v) := [x_0 : x_1]$

the "tautological line bundle of $\mathbb{R}P^1$".



in $\mathbb{R}^2$

disjoint union of those lines



identified

So we get $E \cong$ möbius strip.

More precisely

$U_i := \{[x_0 : x_1] \in \mathbb{R}P^1 \mid x_i \neq 0\}, i = 0, 1$

$\varphi_i : E_{U_i} \longrightarrow U_i \times \mathbb{R}$

$\varphi_i([x_0 : x_1], \lambda(x_0, x_1)) := \left([x_0 : x_1], \lambda x_i \sqrt{1 + \left(\frac{x_{1-i}}{x_i}\right)^2}\right)$

On $U_0 \cap U_1$

$$(\varphi_1)_{[x_0 : x_1]} \circ (\varphi_0)^{-1}_{[x_0 : x_1]}(s) = \begin{cases} s, & \text{if } x_0 x_1 > 0 \\ -s, & \text{if } x_0 x_1 < 0 \end{cases}$$

We get $(p, E, B, \Phi) = (p, E, P\mathbb{R}^1$, completion of $(\varphi_0, U_0), (\varphi_1, U_1))$

Def 6.3 A vector bundle (p, E, B, Φ) is called C^r, $0 \leq r \leq \omega$, if B is a C^r manifold and we have fixed a C^r-structure, i.e. a max C^r-vector bundle atlas $\Phi_{(r)}$ in Φ.

[C^r-vector bundle atlas: ~~Atl~~ All transition maps are C^r.]

We write $(p, E, B, \Phi_{(r)})$ instead of (p, E, B, Φ).

<u>Example 6.4:</u> Both examples in Ex 6.2 are C^ω-vector bundles.

<u>Def 6.5:</u> (i) Let $\xi_i = (p_i, E_i, B_i, \Phi_i)$, $i = 0, 1$, be vector bundles. A map $F: E_0 \to E_1$ is called a <u>fibre map</u> from ξ_0 to ξ_1 (we write $F: \xi_0 \to \xi_1$) if there exists a map $f: B_0 \to B_1$ s.t.

$$\begin{tikzcd} E_0 \arrow[r, "F"] \arrow[d, "p_0"'] \arrow[dr, phantom, "G"] & E_1 \arrow[d, "p_1"] \\ B_0 \arrow[r, "f"] & B_1 \end{tikzcd}$$

We say <u>F covers f</u>.

(ii) A fibre map $F: \xi_0 \to \xi_1$ is called a <u>morphism</u> of vector bundles if

The induced maps

$$\xi_{0x} \cancel{} E_{0x} \xrightarrow{F_x} E_{1,f(x)}$$

are linear for all $x \in B_0$.
A morphism $F: \xi_0 \longrightarrow \xi_1$ is called a monomorphism (epimorphism, bimorphism = vector bundle map) if for all $x \in B_0$ F_x is injective (surjective, bijective)

(iii) A bimorphism $F: \xi_0 \longrightarrow \xi_1$ covering a homeomorphism $f: B_0 \to B_1$ is called an <u>equivalence</u>

(iv) We call $F: \xi_0 \longrightarrow \xi_1$ an <u>isomorphism</u> if $B_0 = B_1 =: B$ and F is an ~~iso~~ equivalence covering id_B. We write "$\cong$".

(v) Def (i) – (iv) make sense for C^r-structures.
In (v) [(iv)] we write "$\cong_r$".

Def 6.6. A C^r-vb ξ is called <u>C^r-trivial</u> if it is C^r-isomorphic to ε_B^n. The corresponding isomorphisms are called C^r-<u>trivializations</u>.

<u>Example 6.7:</u> (i) S^1 has no global C^ω-chart. But $TS^1 \to S^1$ is C^ω-trivial.

$$\begin{tikzcd} TS^1 \arrow[rr, "F"] \arrow[dr, "p"'] & & S^1 \times \mathbb{R} \arrow[dl] \\ & S^1 & \end{tikzcd}$$

$$F(P=(x,y),\ \lambda(-y,x)) := (P, \lambda)$$

<u>Claim</u>: (F, S^1) is a global C^ω-vector bundle chart of $TS^1 \to S^1$.

<u>Proof</u>: For a C^ω-chart (φ, U) of S^1 we obtain a C^ω-vector bundle chart of $TS^1 \xrightarrow{p} S^1$: (φ_{TS^1}, U)

To show

$$U \times \mathbb{R} \xrightarrow{(\varphi_{TU})^{-1}} TU \xrightarrow{F} S^1 \times \mathbb{R}$$

is C^ω. $(F \circ (\varphi_{TU})^{-1})(p, \lambda)$

$$\underset{\uparrow}{=} F\left(\lambda \frac{\partial}{\partial y}(p)\right) \underset{\uparrow}{=} F\left(p, \frac{\lambda}{\sqrt{1-y^2}}(-y, x)\right)$$

(For $(\varphi, U) = (\varphi_{x,+}, U_{x,+})$) $\qquad \frac{\partial}{\partial y}(p) \hat{=} \left(\frac{-y}{\sqrt{1-y^2}}, 1\right)$

$$= \left(p, \frac{\lambda}{\sqrt{1-y^2}}\right)$$

$\Rightarrow$ $F \circ (\varphi_{TU})^{-1}$ is C^ω and has a C^ω inverse

$$(p, \mu) \longmapsto (p, \mu\sqrt{1-y^2}).$$

Analogously on other charts. $\square$

(ii) The tautological C^ω-vector bundle of $\mathbb{R}P^1$ is not trivial (see Ex. 1.29)

$\to$ End of lecture 19.05.2023

Convention 6.8.:

From now on we only consider smooth vector bundles.