Pf: f is continuous, so $f^{-1}(A)$ is closed $\forall A \subseteq \mathbb{R}$ closed.

To show $\forall A \subseteq \mathbb{R}$ bounded: $f^{-1}(A)$ is bounded.

$f^{-1}([-a, a]) = [0, \sqrt{a}]$ is bounded $\square$

End of Lecture 12/05/2023

Def. 5.35: Let M^m and N^m be connected oriented mf, $\partial M = \partial N = \emptyset$ and $f \in C^\infty(M, N)$ proper.
As in Def. 5.28 using Thm. 5.10 there exists
$$\deg(f) \in \mathbb{R}$$
such that
$$f^*[\omega_N]_c = [\omega_M]_c \cdot \deg(f)$$
for $\omega_N \in \Omega^m_c(N)$ and $\omega_M \in \Omega^m_c(M)$
with
$$\int_N \omega_N = 1 = \int_M \omega_M$$

Remark 5.36: Thm. 5.29 holds for proper maps betw. connected oriented

mf. without boundary.
In particular $\deg(f) \in \mathbb{Z}$.

Example 5.37: Let M be a connected closed oriented 2-~~mf~~ submf. of $\mathbb{R}^3$.

(The way how it is embedded in $\mathbb{R}^3$ defines a Riemannian metric on M)

We define $M \xrightarrow{\vec{n}} S^2$

$p \in M$, $v_1, v_2 \in T_pM \subseteq T_p\mathbb{R}^3 \simeq \mathbb{R}^3$ oriented basis

$$\vec{n}(p) := \frac{v_1 \times v_2}{|v_1 \times v_2|_2}$$ ("vector product")

$\vec{n}(p)$ is normal to $T_p(M)$ and $T_{\vec{n}(p)} S^2$

So they identify with the same sub-vector space of $\mathbb{R}^3$.

(move those affine planes to the origin.)

So we get

$$\begin{tikzcd} T_pM \arrow[r, "d\vec{n}"] \arrow[dr, "S"'] & T_{\vec{n}(p)}(S^2) \arrow[d, "\cong"] \\ & T_pM \end{tikzcd}$$

$\vec{n}$ is called the Gauß map of $M \subseteq \mathbb{R}^3$.
S is called the Weingarten map ~~operator~~ (or shape operator).

- The eigenvalues of S are called principal curvatures of M at p
- The determinant of S is called the Gauss curvature of M at p, denoted by $K(p)$

Explicitely: If locally around p

we have parametrization

$$(x,y) \longmapsto (x,y,f(x,y))$$

Then $$\vec{n}(p) = \left(\underbrace{\begin{pmatrix}1\\0\\ \frac{\partial f}{\partial x}\end{pmatrix}}_{\frac{\partial}{\partial x}} \times \underbrace{\begin{pmatrix}0\\1\\ \frac{\partial f}{\partial y}\end{pmatrix}}_{\frac{\partial}{\partial y}} \right) \cdot \frac{1}{|\ \ |_2}$$

$$\doteq \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} \frac{1}{|\ \ |_2}$$

$$= \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{pmatrix} \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}$$

Compute Jac$(d\vec{n})$ w.r.t. $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$

We get $$\det S = \frac{\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}$$

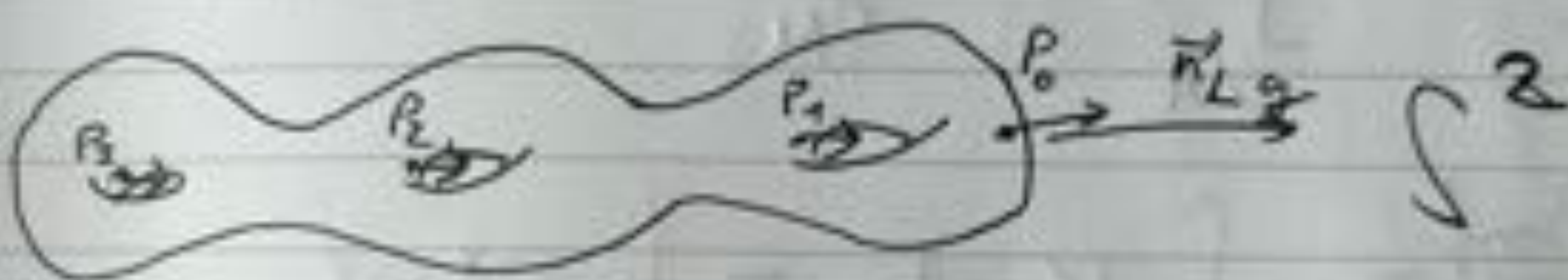
<u>Question</u>: What is the degree of $\vec{n}_g$?

for σ_{S^2} pointing inwards (outwards if σ_g does

Answer: $\deg(\vec{n}) = \frac{\chi(M)}{2}$

Proof: $\vec{n}$ is homotopic to

$L_g \xrightarrow{\vec{n}_{L_g}} S^2$ or $\vec{n}_{-L_g}$.



$Q = (1,0,0) \in S^2 \quad (\vec{n}_{L_g})^{-1}(Q) = \{P_0, P_1, \dots, P_g\}$

$$\Rightarrow \deg(\vec{n}) = \deg(\vec{n}_{L_g}) = 1 - g$$

$$= \frac{\chi(M)}{2} \qquad \square$$

Exercise: Let g_{S^2} be the metric induced from $\mathbb{R}^3$. Analogously g_M.

Then $\vec{n}^* vol_{g_{S^3}} = K \cdot vol_{g_M}$

Theorem 5.38: (Gauß - Bonnet)

Under the conditions of Example 5.37. we have

$$\int_M K \, vol_{g_M} = 2\pi \chi(M)$$

Proof: $\int_M K \, vol_{g_M} = \int_M \vec{n}^* vol_{g_{S^2}}$

$$= deg(\vec{n}) \int_{S^2} vol_{g_{S^2}} = \frac{\chi(M)}{2} \cdot vol_{g_{S^2}}(S^2)$$

$$= \frac{\chi(M)}{2} 4\pi = 2\pi \chi(M) \quad \square$$

Theorem 5.38: (Gauß - Bonnet)

Under the conditions of Example 5.37. we have

$$\int_M K \, vol_{g_M} = 2\pi \, \chi(M)$$

Proof:

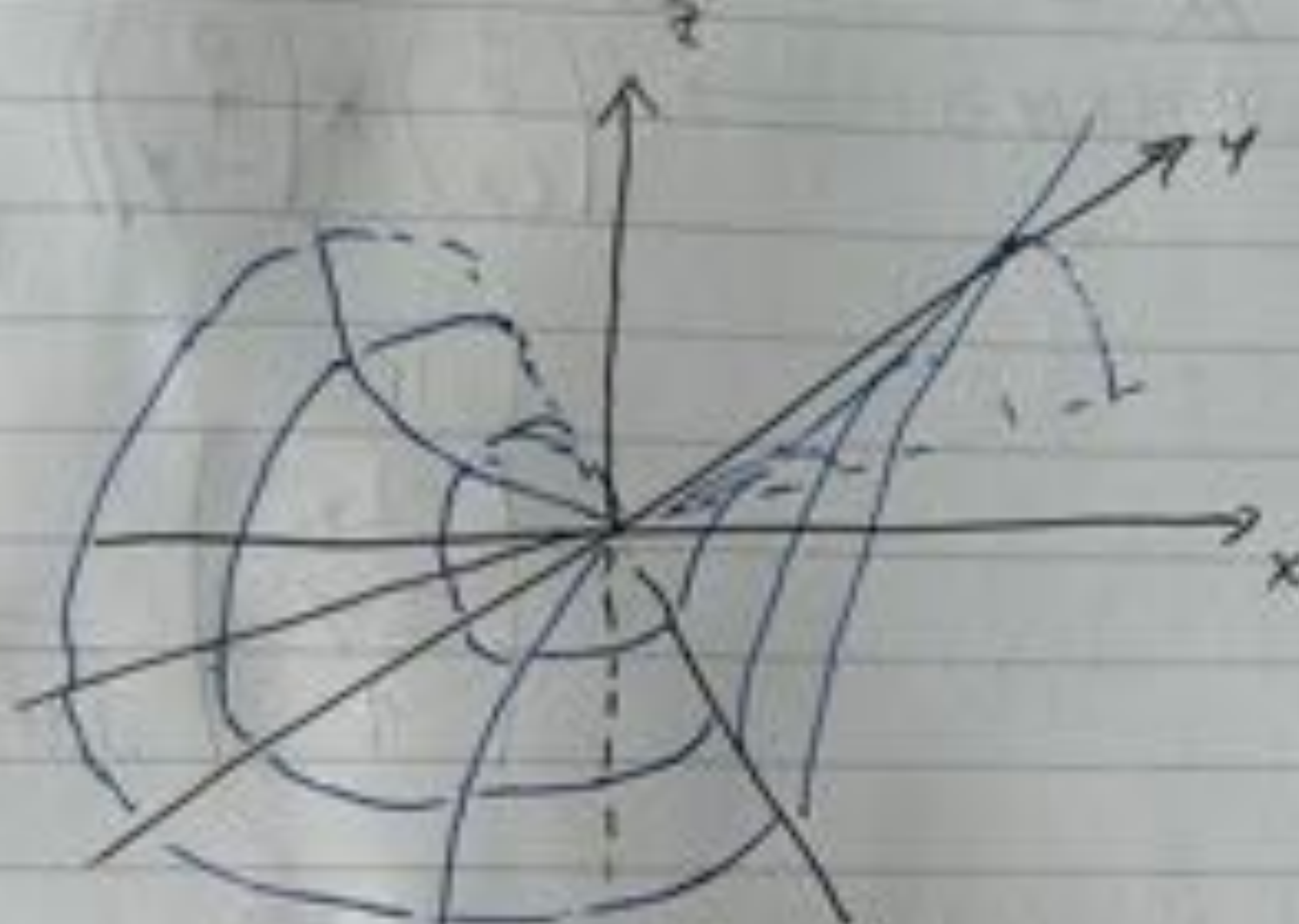
$$\int_M K \, vol_{g_M} = \int_M \vec{n}^* vol_{g_{S^2}}$$

$$= \deg(\vec{n}) \int_{S^2} vol_{g_{S^2}} = \frac{\chi(M)}{2} \cdot vol_{g_{S^2}}(S^2)$$

$$= \frac{\chi(M)}{2} 4\pi = 2\pi \, \chi(M) \quad \square$$

"Hyperbolic paraboloid"

Ex 5.39: $M = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$



A saddle.

Compute $K(\underline{0})$ w.r.t. the induced Riemannian metric g_M?

We have a global chart $M \xrightarrow{\varphi} \mathbb{R}^2$
$(x,y,z) \longmapsto (x,y)$

so $T_pM = \mathbb{R}\begin{pmatrix}1\\0\\2x\end{pmatrix} + \mathbb{R}\begin{pmatrix}0\\1\\-2y\end{pmatrix}$, for $p = (x,y,z)$.

We consider the orientation by the upward pointing normal vector field, i.e.

$$\sigma(p) = \left[\begin{pmatrix}1\\0\\2x\end{pmatrix}, \begin{pmatrix}0\\1\\-2y\end{pmatrix}\right]$$

27.1

The Gauß-map is given by

$$\vec{n} : M \longrightarrow S^2$$

$$p = (x,y,z) \longmapsto \left(\begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2y \end{pmatrix} \right) \cdot \frac{1}{|\ldots|_2}$$

$$\parallel$$

$$\begin{pmatrix} -2x \\ 2y \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{1+4(x^2+y^2)}}$$

$$d\vec{n}\left(\frac{\partial}{\partial x}\right) = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{1+4(x^2+y^2)}} + \begin{pmatrix} -2x \\ 2y \\ 1 \end{pmatrix} \frac{-4x}{\sqrt{1+4(x^2+y^2)}^3}$$

$$= \frac{-2}{\sqrt{1+4(x^2+y^2)}^3} \left((1+4y^2)\frac{\partial}{\partial x} + 4xy\frac{\partial}{\partial y} \right)$$

after identification of T_pM with $T_{\vec{n}(p)}S^2$.

$$d\vec{n}\left(\frac{\partial}{\partial y}\right) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \frac{1}{\sqrt{1+4(x^2+y^2)}} + \begin{pmatrix} -2x \\ 2y \\ 1 \end{pmatrix} \frac{-4y}{\sqrt{1+4(x^2+y^2)}^3}$$

$$= \frac{2}{\sqrt{1+4(x^2+y^2)}^3} \left((1+4x^2)\frac{\partial}{\partial y} + 4xy\frac{\partial}{\partial x} \right)$$

So the transformation matrix for the Weingarten map in terms of $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is given by

$$\mathcal{M}_{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}(S) = \begin{pmatrix} -(1+4y^2) & 4xy \\ -4xy & (1+4x^2) \end{pmatrix} \frac{2}{\sqrt{1+4(x^2+y^2)}^{\,3}}$$

We obtain:

$$K(P) = \det\left(\mathcal{M}_{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}(S)\right)_{\text{at } P}$$

$$= \frac{4}{(1+4(x^2+y^2))^3} \cdot (-1) \cdot (1+4x^2+4y^2)$$

$$= \frac{-4}{(1+4(x^2+y^2))^2}$$

The principal curvatures at $P = \underline{0}$:

$$\chi_{\mathcal{M}(S_{\underline{0}})}(T) = \chi_{\begin{pmatrix} -2 & \\ & 2 \end{pmatrix}}(T) = T^2 - 4 = (T-2)(T+2)$$

The eigenvalues are 2 and -2.

27.3

(b) $M = S_r^2 = \{(x,y,z) \mid x^2+y^2+z^2 = r^2\}$, $r > 0$.

outwards orientation.

$\vec{n}(p) = \frac{1}{r}(x,y,z)$ for $p = (x,y,z)$

$$\Rightarrow M_{(v_1,v_2)}(S_p) = \begin{pmatrix} \frac{1}{r} & \\ & \frac{1}{r} \end{pmatrix}$$

for any oriented basis of $T_p(S_r^2)$ (v_1, v_2).

The principal curvatures are $\frac{1}{r}, \frac{1}{r}$ everywhere and

$$K_{S_r^2}(p) = \frac{1}{r^2}.$$

End of Lecture 16.05.2023