The de Rham cohomology with compact support allows to define the degree of a map.

Def 5.28: Let M^m, N^m be connected closed oriented mf and $f \in C^\infty(M,N)$

Consider the diagramm

$$
\begin{tikzcd}
{[\omega_N]} & H^m_{dR}(N) \arrow[rr, dashed, "f^*"] & & H^m_{dR}(M) & {[\omega_M]} \\
1 \arrow[u, mapsto] & \mathbb{R} \arrow[u] \arrow[rr, dashed] \arrow[urr, phantom, "\circlearrowleft"] & & \mathbb{R} \arrow[u] & 1 \arrow[u, mapsto] \\
 & x \arrow[rr, mapsto] & & Dx &
\end{tikzcd}
$$

with $\int_N \omega_N = 1 = \int_M \omega_M$

Then "$\dashrightarrow$" is a multiplication with an element D of $\mathbb{R}$.

D is called the degree of f.

<u>Theorem 5.29</u>: f, M, N as in Def 5.28.

Let Q be a regular value of f. ~~...~~

Let E be the number of elements of $f^{-1}(Q)$ counted with a sign according to $\det(df)$.

Then
$$\forall \omega \in \Omega^m(N): \int_M f^*\omega = E \int_N \omega$$

<u>Remark 5.30</u>:

(a) The sign is meant as follows.

Suppose $f(P) = Q$, $(\varphi, \overset{P}{\hat{U}}), (\psi, \overset{Q}{\hat{V}})$ oriented charts.
We count P with positive (negative) sign if

$$\det\left(D(\psi \circ f \circ \varphi^{-1})(\varphi(P))\right)$$

is positive (negative)

(b) Take $\omega = \omega_N \Rightarrow \mathbb{Z} \ni E = \deg(f)$.

Example 5.31:

(i) $f : S^1 \longrightarrow S^1$
$e^{i\theta} \longmapsto e^{im\theta}$

$\Rightarrow f^*: H^1(S^1) \longrightarrow H^1(S^1)$
$d\theta \longmapsto m\,d\theta$

$\Rightarrow \deg(f) = m$

$\underset{e^{i\theta}}{f^{-1}(1)} = \{ e^{i\frac{2\pi j}{m}} \mid j = 0, 1, \dots, m-1 \}$

all counted with sign +1.

m=5 

(ii) $S^3 \xrightarrow{\pi} \mathbb{RP}^3$

$\cancel{e^{i\theta}} \;\cancel{\longmapsto}\; \cancel{[e^{i\theta}]}$

$\underline{x} \longmapsto [\underline{x}] = [x_0 : x_1 : x_2 : x_3]$

Take $Q = [1; 0; 0; 0]$ $\pi^{-1}(Q) = \{(1,0,0,0), (-1,0,0,0)\}$

We take on S^3 the orientation induced from the outwards pointing normal vectorfield.

and on $\mathbb{R}P^3$ the one s.t. locally around $(1,0,0,0)$ π preserves the orientations.

<u>Claim</u>: The map $\underline{x} \longmapsto -\underline{x}$ on S^3 preserves the orientation

<u>Proof</u>: Consider $\Phi : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$\Phi(\underline{x}) := -\underline{x}$, Take $R \in \mathbb{R}^4$

Then $d\Phi_R\left(\frac{\partial}{\partial x_i}(R)\right) = -\frac{\partial}{\partial x_i}(\Phi(R))$

$\Rightarrow$ The Jacobi matrix is $\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ and has determinat $1 > 0$.

$\Longrightarrow$ Φ perserves the orientation of $\mathbb{R}^4$ and $d\Phi$ preserves the normal outwards pointing v.f. of S^3.

$\Longrightarrow$ $\Phi|_{S^3}$ preserves the orientation of S^3 $\square$

claim $\Rightarrow$ $d\pi_{(-1,0,0,0)}$ has positive determinant.

$\Rightarrow \deg \pi = 2$

(iii) M as in Def 5.28., $f \in \mathrm{Diff}^{\sim}(M)$

$$\deg(f) = \begin{cases} +1, & \text{if } f \text{ preserves orientation} \\ -1, & \text{else.} \end{cases}$$

(iv) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

<u>Proof of Thm 5.29:</u> Part 1 (local observation around Q)

<u>Case 1:</u> $f^{-1}(Q) = \emptyset \Rightarrow \exists\, V \subseteq N$ open around Q: $\Rightarrow \forall\, \omega \in \Omega_c^m(V)$: $f^*(\omega) = 0$.

$$\Rightarrow \int_M f^*\omega = 0 \underset{\uparrow}{=} E \cdot \int_N \omega .$$

$E = 0$, because $f^{-1}(Q) = \emptyset$

Case 2: Take $p \in f^{-1}(Q)$ and

$V \ni Q$ and $U \ni p$ open s.t.
$f: U \xrightarrow{\sim} V$ diffeomorphism.

$\Rightarrow \forall\, \omega \in \Omega^m_c(V)$:

(*) $$\int_U f^*\omega = \pm \int_V \omega$$

preserving/
not-"-
orientation

Now $f^{-1}(Q) = \{p_1, \dots, p_\ell\}$
$\varepsilon_1, \dots, \varepsilon_\ell \leftarrow$ signs.
is finite, because M is compact and Q is a regular value.

(*) $\Rightarrow$ $$\int_M f^*\omega = \sum_{i=1}^{\ell} \int_{U_i} (f^*\omega)|_{U_i}$$

V small enough around Q, s.t. $f^{-1}(V) = \dot{\bigcup}_{i=1}^{\ell} U_i$ with $f: U_i \xrightarrow{\sim} V$.

$$= \sum_{i=1}^{\ell} \varepsilon_i \int_V \omega$$

$$= E \int_V \omega. \ \forall \omega \in \Omega^m_c$$

Part 2: $\omega \in \Omega^m_c(N)$ is cohomologous to some $\omega_0 \in \Omega^m_c(V)$, V taken as in Case 1/2 of Part 1.

$\Rightarrow$ $f^*\omega$ and $f^*\omega_0$ are cohomologous and

$$\int_M f^*\omega = \int_{f^{-1}(V)} f^*\omega_0 = E \int_V \omega_0 = E \int_N \omega .$$

$\square$

Remark 5.32: We don't need M to be compact to define the degree of a map if we assume that the map is proper.

Def 5.33: $f \in C^{\infty}(M, N)$ is called proper if $\forall K \subseteq N$ compact the preimage $f^{-1}(K)$ is compact.

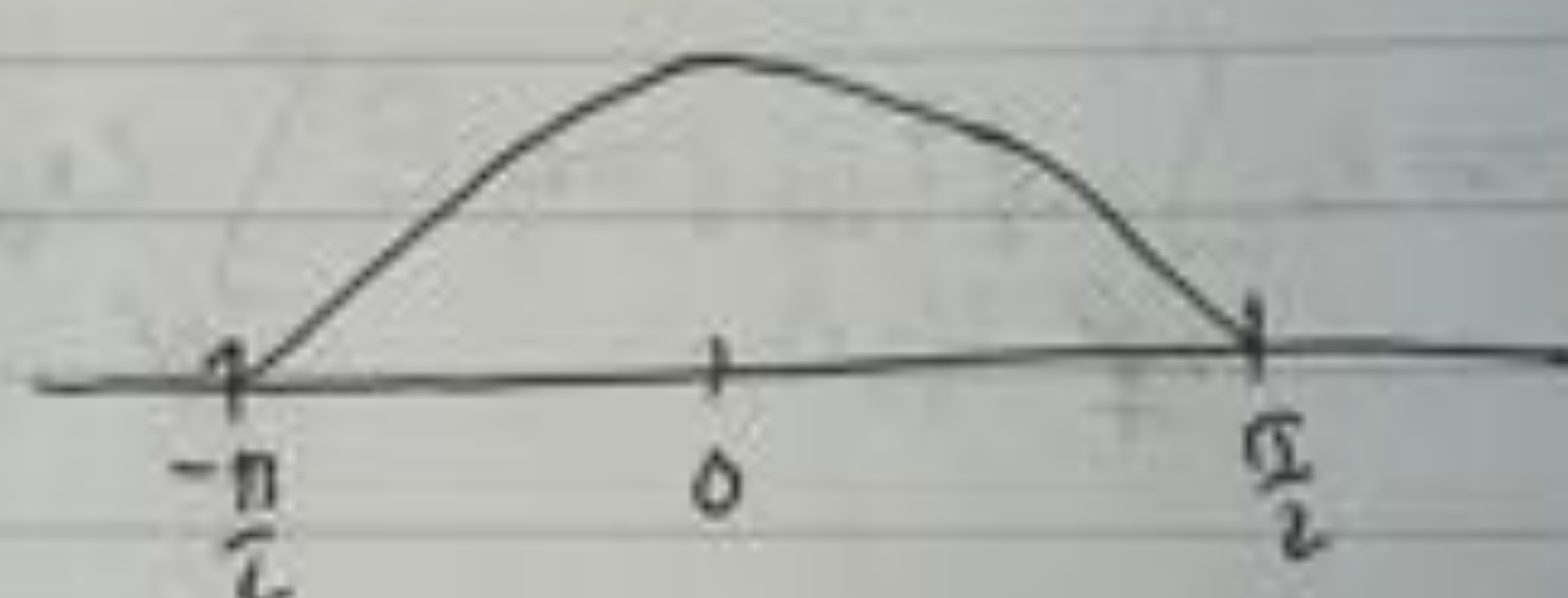
Example 5.34:

(i) $\mathrm{id} : M \longrightarrow M$ is proper.

(ii) $f : M \longrightarrow \{p\}$ is proper iff M is compact.

(iii) $f : \underbrace{[-\frac{\pi}{2}, \frac{\pi}{2}]}_{=:M} \longrightarrow \mathbb{R}$

$f(x) = \cos(x)$



$f\big|_{M \setminus \{p\}}$ is not proper for any $p \in M$, because

$f^{-1}([0,1]) \cap (M \setminus \{p\})$

$= [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{p\}$ is not compact

(iv) $f : \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f(x) = x^2$

Then f is proper.

Pf: f is continuous, so $f^{-1}(A)$ is closed $\forall A \subseteq \mathbb{R}$ closed.

To show $\forall A \subseteq \mathbb{R}$ bounded: $f^{-1}(A)$ is bounded.

$$f^{-1}([-a, a]) = [0, \sqrt{a}] \text{ is bounded}$$

End of Lecture 12/05/2023 ☐

Def. 5.35: Let M^m and N^m be connected oriented mfs $\partial M = \partial N = \emptyset$ and $f \in C^\infty(M, N)$ proper.
As in Def. 5.28 using Thm. 5.10 there exists

$$\deg(f) \in \mathbb{R}$$

such that

$$f^*[\omega_N]_c = [\omega_M]_c \cdot \deg(f)$$

for $\omega_N \in \Omega_c^m(N)$ and $\omega_M \in \Omega_c^m(M)$

with $$\int_N \omega_N = 1 = \int_M \omega_M.$$

Remark 5.36: Thm. 5.29 holds for proper maps betw. connected oriented