(I) $0 \longrightarrow (H_c^m(N\cup V))^* \longrightarrow H_c^m(N)^* \oplus H_c^m(V)^* \hookrightarrow H_c^m(N\cap V)^* \xrightarrow{\delta^*} \cdots$

$\cdots \longrightarrow H_c^0(N\cup V)^* \longrightarrow H_c^0(N)^* \oplus H_c^0(V)^* \to H_c^0(N\cap V)^* \to$

(II) $0 \longrightarrow H^0(N\cup V) \to H^0(N) \oplus H^0(V) \to H^0(N\cap V) \xrightarrow{\partial} \cdots$

$\cdots \to H^m(N\cup V) \to H^m(N) \oplus H^m(V) \to H^m(N\cap V) \to 0$

The form (*) connects both sequences.

$$\begin{array}{ccc}
H^k(N\cup V) & \xrightarrow{[\varphi] \mapsto \int_{N\cup V} -\wedge\varphi} & (H_c^{m-k}(N\cup V))^* \\
\downarrow & & \downarrow \\
H^k(N) \oplus H^k(V) & \xrightarrow{([\varphi],[\tau]) \mapsto (\int_N -\wedge\varphi,\ \int_V -\wedge\tau)} & (H_c^{k}(N))^* \oplus (H_c^{m-k}(V))^* \\
\downarrow & & \downarrow \\
H^k(N\cap V) & \xrightarrow{[\varphi] \mapsto \int_{N\cap V} -\wedge\varphi} & (H^{m-k}(N\cap V))^* \\
\delta \downarrow & & \delta^* \downarrow \\
H^{k+1}(N\cup V) & \xrightarrow{[\varphi] \mapsto \int_{N\cup V} -\wedge\varphi} & (H^{m-k-1}(N\cup V))^*
\end{array}$$

up to some signs

If we show that this diagram commutes then the theorem follows from Lemma 5.15, the 5 lemma.

$$\begin{tikzcd}
\cdot \arrow[r] & \cdot \arrow[r] & (H^k(N\cup V))^* \arrow[r] & \cdot \arrow[r] & \cdot \\
\cdot \arrow[r] \arrow[u, "\wr"] & \cdot \arrow[r] \arrow[u, "\wr"] & H^{m-k}(N\cup V) \arrow[r] \arrow[u] & \cdot \arrow[r] \arrow[u, "\wr"] & \cdot \arrow[u, "\wr"]
\end{tikzcd}$$

The middle arrow must be an isomorphism.

The proof of the commutativity of the diagram consists of two cases.

Case 1:

$$\begin{tikzcd}
H^k(M) \arrow[r] \arrow[d] & (H_c^{m-k}(M))^* \arrow[d] \\
H^k(N) \arrow[r] & (H_c^{m-k}(N))^*
\end{tikzcd}$$

Case 2:

$$\begin{tikzcd}
H^k(N\cap V) \arrow[r] \arrow[d, "\delta"'] & (H_c^{m-k}(N\cap V))^* \arrow[d, "\delta^*"] \\
H^{k+1}(N\cup V) \arrow[r] & (H_c^{m-k-1}(N\cup V))^*
\end{tikzcd}$$

Proof of Case 1:

$$\begin{tikzcd}
{[1]} \arrow[r, mapsto] \arrow[d] & \int_M - \wedge 1 \arrow[d] \\
{[1|_N]} \arrow[r, mapsto, dashed] & \left(\int_M - \wedge 1\right) \circ \pi
\end{tikzcd}$$

$$\pi : H_c^{(m-k)}(N) \longrightarrow H_c^{m-k}(M).$$

and for $[\omega]_c \in H_c^{(m-k)}(N)$ we have

$$\pi([\omega]_c) = [\tilde{\omega}]_c \in H_c^{m-k}(M)$$

where $\tilde{\omega}$ is the extension of ω by 0.

and

$$\int_M \tilde{\omega} \wedge 1 \underset{\substack{\uparrow \\ \operatorname{supp}(\tilde{\omega}) \subseteq N}}{=} \int_N \omega \wedge 1|_N$$

Which is "$\mapsto$".

Proof of Case 2: Take a partition of unity for $(N, V) : (\lambda, \mu)$.

Take $[\rho] \in H^k(N \cap V)$.

Then $\delta([\rho]) = [\tilde{\rho}] \in H^{k+1}(M)$
with
$$\tilde{\rho}|_N = d\rho_1, \ \tilde{\rho}|_V = d\rho_2 \text{ s.t. } \rho_1|_{N\cap V} - \rho_2|_{N\cap V} = \rho$$

$$\begin{array}{ccc} [\rho] \in H^k(N \cap V) & \longmapsto & F_\rho := \int_{N\cap V} - \wedge \rho \in (H_c^{m-k}(N\cap V))^* \\ \downarrow & & \downarrow \\ [\tilde{\rho}] \in H^{k+1}(M) & & F_\rho \circ \delta \in (H_c^{m-k-1}(M))^* \end{array}$$

with $\delta : H_c^{m-k-1}(M) \to H_c^{m-k}(N \cap V)$

is given by:

$$\delta([\omega]_c) = [\omega_0]_c \in H_c^{m-k}(N \cap V)$$

s.t. $(i_{N\cap V}^M)_* \omega_0 = d\omega_1$, $(i_{N\cap V}^V)_* \omega_0 = d\omega_2$

s.t. $\omega_1 - \omega_2 = \omega$.

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We have to show:

$$\int_M \omega \wedge \tilde{\varsigma} = \int_{N\cap V} \omega_0 \wedge \varsigma$$

Pf: $\displaystyle\int_M \omega \wedge \tilde{\varsigma} = \int_M (\omega_1 - \omega_2) \wedge \tilde{\varsigma}$

$$= \int_N \omega_1 \wedge \tilde{\varsigma} - \int_V \omega_2 \wedge \tilde{\varsigma}$$

$$= \int_N \omega_1 \wedge d\varsigma_1 - \int_V \omega_2 \wedge d\varsigma_2$$

$$= (-1)^{m-k} \left(\int_N (d\omega_1) \wedge \varsigma_1 - \int_V (d\omega_2) \wedge \varsigma_2 \right)$$

$$\underset{\substack{\uparrow \\ d\omega_1, d\omega_2}}{=} (-1)^{m-k} \int_{N\cap V} \omega_0 \wedge (\varsigma_1|_{N\cap V} - \varsigma_2|_{N\cap V})$$

have supp in $N\cap V$

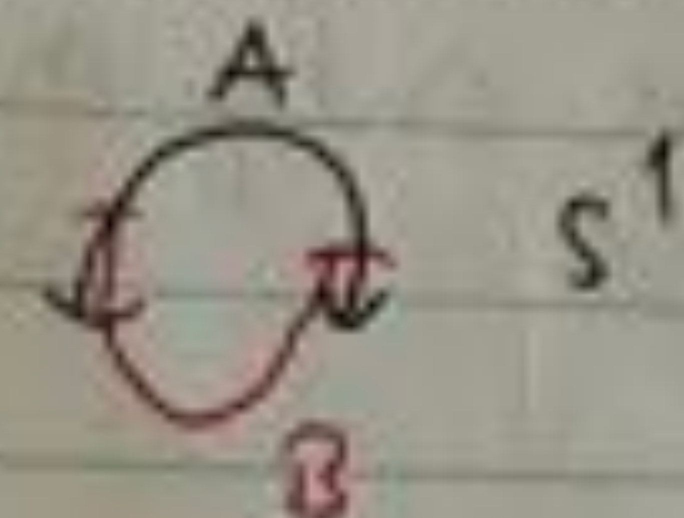
and $d\omega_1|_{N\cap V} = d\omega_2|_{N\cap V} = \omega_0$

$$= (-1)^{m-k} \int_{N\cap V} \omega_0 \wedge \varsigma \qquad \square$$

<u>Example 5.18:</u> $M = S^2 \times S^1$. Compute $H^k_{dR}(M)$ for $0 \le k \le 3$.

At first note that M is orientable because S^2 and S^1 are.
(Exercise! Hint: Find a nowhere vanishing 3-form on M.)

$S^1 = A \cup B$



$U := S^2 \times A$, $V := S^2 \times B$

Mayer-Vietoris sequence

$$0 \to \underbrace{H^0(U \cup V)}_{\mathbb{R}} \to \underbrace{H^0(U) \oplus H^0(V)}_{\mathbb{R}^2} \to \underbrace{H^0(U \cap V)}_{\mathbb{R}^2} \to H^1(M) \to 0$$

because U and V are homotopic to S^2

$\Rightarrow$ $H^1(M) \simeq \mathbb{R}$.

Poincaré duality

$\Rightarrow$ $H^k_{dR}(M) \simeq \mathbb{R}$, $k = 0, \dots, 3$.

<u>Example 5.19:</u> Let M be a closed connected orientable 3-manifold. Then $H^0_{dR}(M) \simeq H^3_{dR}(M) \simeq \mathbb{R}$

and $H^1_{dR}(M) \simeq H^2_{dR}(M)$.

Then we only need to compute $H^1_{dR}(M)$.

<u>Question:</u> Given $h_1 \in \mathbb{N}_0$
Does there exists a closed connected orientable 3-mf. M such that $H^1_{dR}(M) \simeq \mathbb{R}^{h_1}$?

<u>Ex:</u> $h_1 = 0$: $M = S^3$, or $M = \mathbb{R}P^3$

$h_1 > 0$?

<u>Def 5.20:</u> (connected sums of mf.)

(1) Let $B_1(\underline{0})$ be the unit ball in $\mathbb{R}^m$, $m \geq 1$. We define

glue: $\underbrace{B_1(\underline{0}) \setminus \overline{B_{\frac{1}{2}}}(\underline{0})}_{=: B_{1,\frac{1}{2}}(\underline{0})} \circlearrowleft \quad B_{1,\frac{1}{2}}(\underline{0})$

via $\mathrm{glue}(\underline{x}) := \left(\frac{1}{2} + 1 - |\underline{x}|_2\right) \frac{\underline{x}}{|\underline{x}|_2}$.

(They are meant to be open in $\mathbb{R}^m$)

(2) Let M and N be m-dim manifolds, $m \geq 1$, both connected.

Let (φ, U) and (ψ, V) be charts on M and N, respectively, such that $\varphi(U) = B_1(\underline{0}) = \psi(V)$

Write $U_{1,\frac{1}{2}} := \varphi^{-1}(B_{1,\frac{1}{2}}(\underline{0}))$

$$V_{1,\frac{1}{2}} = \psi^{-1}(B_{1,\frac{1}{2}}(\underline{0})).$$

We define $M \# N$ to be the mf. obtained as

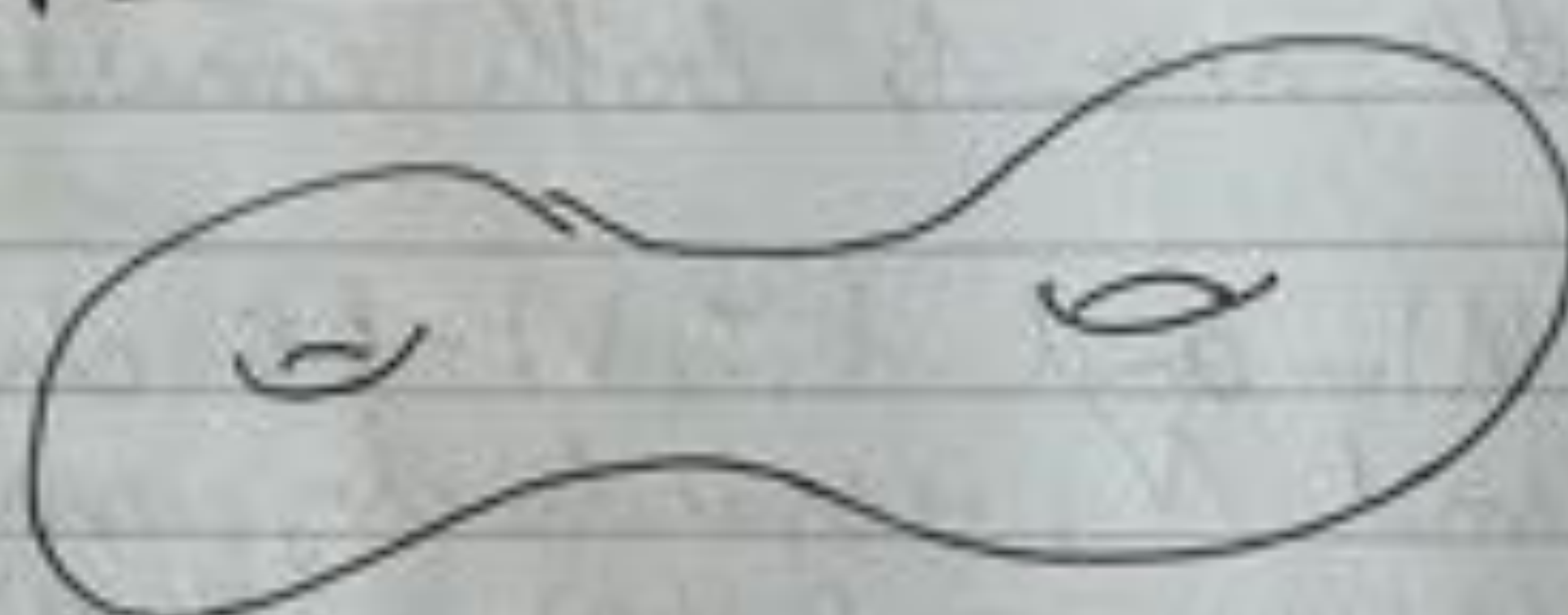
$$M \sqcup N \Big/ \{(P,Q) \mid P \in U_{1,\frac{1}{2}}, Q \in V_{1,\frac{1}{2}}, P = (\varphi^{-1} \circ \mathrm{glue} \circ \psi)(Q)\}$$

The diffeomorphism class does not depend on the choice of (φ, U) and (ψ, V).

$M \# N$ is called the "connected sum of M and N".

Example 5.21:

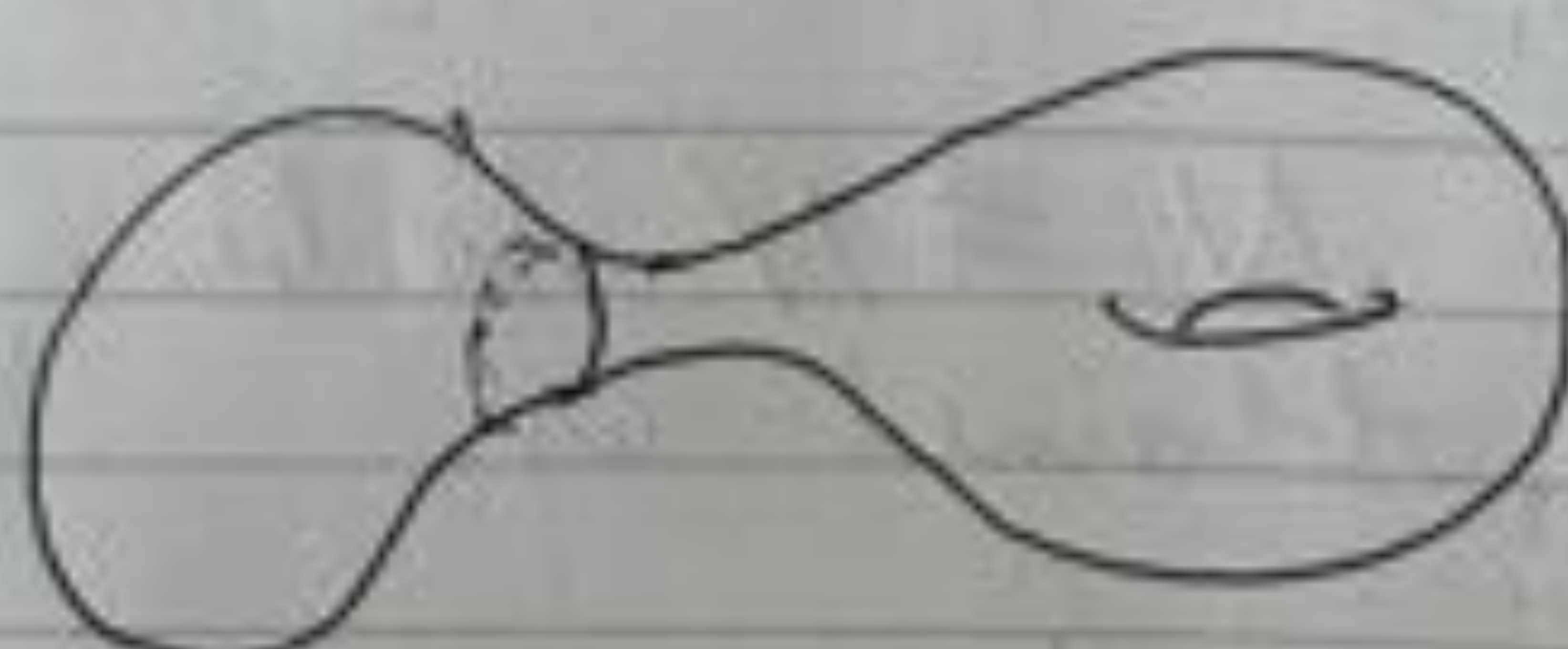
(a)



$T^2 \# T^2$

where $T^2 = S^1 \times S^1$.

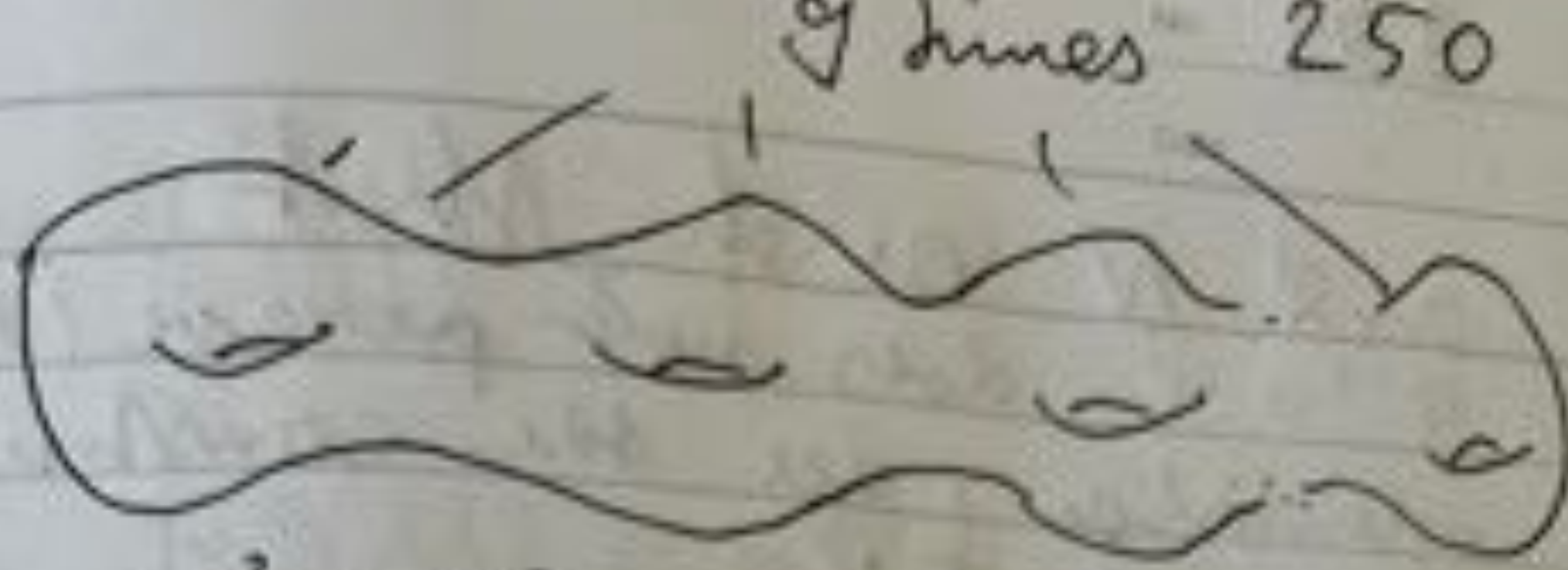
(b)



$$\underset{\uparrow}{B_1(\underline{0})_2} \# T^2 \simeq T^2$$

(open disc)

(c)



$T^2 \# T^2 \# \cdots \# T^2 = T^{2\#g}$

(d) K^2

$K^2 \# T^2$

<u>Lemma 5.22:</u> Let M^m and N^m, $m \geq 1$, be orientable connected mf.
Then
$M \# N$ is orientable.

<u>Proof:</u> Take orientations on M

and N such that
$\varphi^{-1}\circ\psi$ does not preserve the orientation, see the construction of $M \# N$.
Then $\varphi^{-1}\circ \text{glue} \circ \psi \big|_{V_1, V_2}$

preserves the orientations. $\square$

<u>Prop 5.23:</u> Let M^m, N^m, $m \geq 2$

be connected closed mf.
Then

$$H^k(M \# N) \simeq \begin{cases} \mathbb{R}, & k=0 \\ H^k(M) \oplus H^k(N), & 1 \leq k \leq m-1 \end{cases}$$

and $H^m(M \# N) \simeq \begin{cases} \mathbb{R}, & \text{if } M \text{ and } N \text{ are orient} \\ 0, & \text{else.} \end{cases}$

and $H^{m-1}(M \# N) \simeq \begin{cases} H^{m-1}(M) \oplus H^{m-1}(N), & \text{if } M \text{ or } N \text{ is orien} \\ H^{m-1}(M) \oplus H^{m-1}(N) \oplus \mathbb{R}, & \text{else} \end{cases}$

<u>Lemma 5.24:</u> Let M^m, $m \geq 1$,
be a connected mf. and $p \in M$.

Thm (a) $H^k(M) \simeq H^k(M \setminus \{p\})$
for all $1 \leq k \leq m-2$.

(b) $H^0(M) \simeq H^0(M \setminus \{p\})$

(c) $H^{m-1}(M) \simeq H^{m-1}(M \setminus \{p\})$
if M is closed and orientable

(d) $H^{m-1}(M) \oplus \mathbb{R} \simeq H^{m-1}(M \setminus \{p\})$
if M is closed and not orientable, $m \geq 2$.

(e) $H^m(M \setminus \{p\}) = 0$ if M is closed.

Proof: (b) ✓

(a) We get for $1 \leq k \leq m-2$

$$0 \to H^k(M) \to H^k(M \setminus \{p\}) \oplus H^k(p) \to 0$$

in the M-V sequence (also for $k = 1$!), because $H^k(S^{m-1}) = 0$.

for $m \geq 2$:

(c) $0 \to H^{m-1}(M) \to H^{m-1}(M \setminus \{p\}) \xrightarrow{\delta} H^{m-1}(S^{m-1})$

$\hookrightarrow H^m(M) \longrightarrow H^m(M \setminus \{p\}) \longrightarrow 0$

Elements of $\Omega^m(M \setminus \varphi^{-1}(\text{open ball}))$ are exact, for (φ, U) a chart around p,

(Why? We can extend them to M with integral zero.)

Thus $H^m(M \setminus \{p\}) = 0$ and δ is bijective as $H^m(M) \cong \mathbb{R} \cong H^{m-1}(S^{m-1})$.

$\Longrightarrow H^{m-1}(M) \cong H^{m-1}(M \setminus \{p\})$.

(d) $m \geq 2$,

$0 \to H^{m-1}(M) \to H^{m-1}(M \setminus \{p\}) \to \mathbb{R}$

$\hookrightarrow H^m(M) = 0 \to H^m(M \setminus \{p\}) \to 0$

$\Longrightarrow H^{m-1}(M \setminus \{p\}) \cong H^{m-1}(M) \oplus \mathbb{R}$

(e) <u>$m \geq 2$:</u> See (c) and (d).

<u>$m=1$:</u> Similar $\square$

<u>Example 5.25:</u> $H^1_{dR}(T^2 \setminus \{P\}) \simeq H^1_{dR}(T^2)$

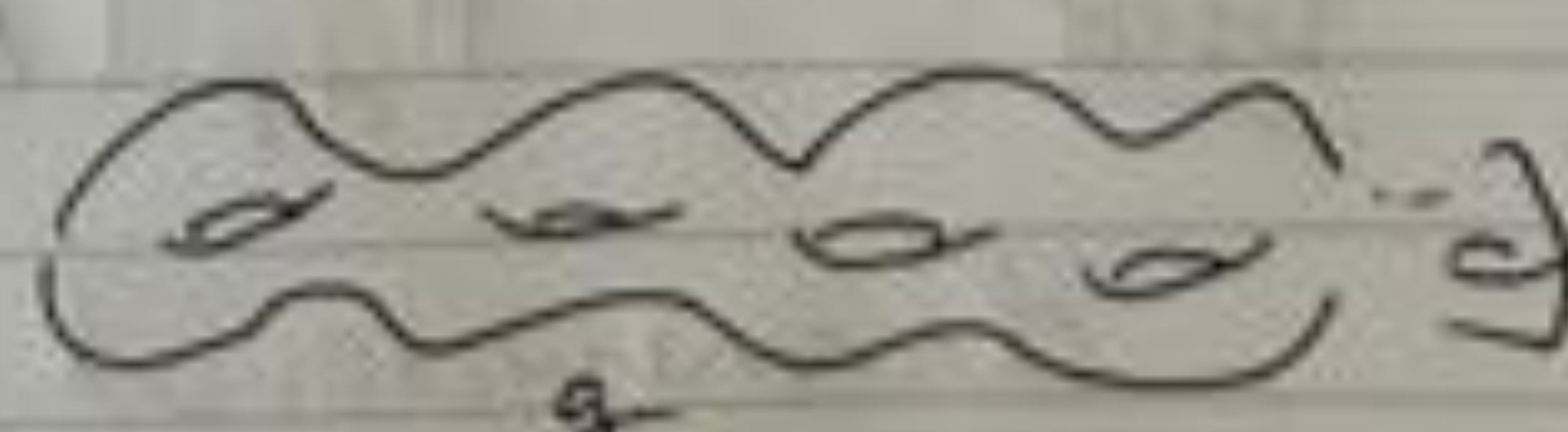
$\simeq H^1_{dR}(S^1 \times S^1) \simeq \mathbb{R}^2$.

Which was an exercise in Problem sheet 9.

<u>Proof of Thm 5.23:</u> Just the Mayer-Vietoris sequence with

$M \setminus \varphi^{-1}(\overline{B}_{\frac{1}{2}}(\underline{0}))$ and $N \setminus \psi^{-1}(\overline{B}_{\frac{1}{2}}(\underline{0}))$.

using Lemma 5.24. $\square$

<u>Example 5.26:</u> L_g 

$$H^1_{dR}(L_g \times S^1) \simeq \mathbb{R} \oplus H^1_{dR}(L_g)$$
$$\simeq \mathbb{R}^{2g+1}$$

and $H^1_{dR}((\Sigma_g \times S^1) \# (S^2 \times S^1))$
$\simeq \mathbb{R}^{2g+1} \oplus \mathbb{R} \simeq \mathbb{R}^{2g+2}$

and $H^1_{dR}(S^3) = 0$.

Thus, for all $h \in \mathbb{N}_0$

$\exists$ connected, orientable, closed 3-mf with $H^1_{dR} \simeq \mathbb{R}^h$.

(See question in Ex. 5.19)

Example 5.27:

$M = (S^2 \times S^1) \# (S^2 \times S^1)$ is not diffeom. to a product of connected closed orientable mf..

Proof: $H^1_{dR}(M) \simeq \mathbb{R}^2$ is even dimensional.

$H^1_{dR}(\Sigma_g \times S^1) \simeq \mathbb{R}^{2g+1}$ odd di[illegible] $\square$

— End of Lecture 9.5.2023.