# Chapter V

## Compactly supported de Rham complex

Def 5.1: Let $M^m$ be a compact manifold. The complex

$$0 \to \Omega_c^0(M) \xrightarrow{d_c^{(0)}} \Omega_c^1(M) \xrightarrow{d_c^{(1)}} \Omega_c^2(M) \to \cdots$$
$$\to \Omega_c^{m-1}(M) \xrightarrow{d} \Omega_c^m(M) \to 0$$

is called the compactly supported de Rham complex of $M$.

For $k \geq 0$ we define the compactly supported de Rham cohomology via

$$H_c^k(M) := \ker(d_c^{(k)}) \big/ \operatorname{im}(d_c^{(k-1)})$$

Theorem 5.2: Let $n \geq 0$ and $k \geq 0$

Then $$H_c^k(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R}, & k = n \\ 0, & k \neq n. \end{cases}$$

Example 5.3:

(a) $n \geq 0$: $\quad H^0_c(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & n=0 \\ 0, & n>0 \end{cases}$

because $\mathbb{R}^0 = \{pt\}$ is compact and a compactly supp fct$^n$ on $\mathbb{R}^n$ if constant has to be zero for $n \geq 1$.

(b) $H^1_c(\mathbb{R}) \cong \mathbb{R}$. Proof:

Stoke's $\Rightarrow$ $\operatorname{im} d^{(0)}_c \subseteq \{\omega \in \Omega^1_c(\mathbb{R}) \mid \int_{\mathbb{R}} \omega = 0\}$

"$\supseteq$" $\omega \in \Omega^1_c(\mathbb{R})$ with $\int_{\mathbb{R}} \omega = 0$

$H^1_{dR}(\mathbb{R}) = 0 \Rightarrow \exists f \in C^\infty(\mathbb{R})\ \omega = df$

Take $[a,b]$ s.t. $\operatorname{supp}(\omega) \subseteq ]a,b[$.
Then $0 = \int_{\mathbb{R}} \omega = \int_{[a,b]} \omega = f(b) - f(a)$

$\Rightarrow \exists c \in \mathbb{R}\ \exists q > 0\ \forall x \in \mathbb{R}$ with $|x| \geq q$

$f(x) = c$.

$\Rightarrow g := f - c\, id_{\mathbb{R}} \in \Omega^0_c(\mathbb{R}),\ dg = \omega$.

The map $H^1_c(\mathbb{R}) \longleftrightarrow \mathbb{R}$
$[\omega] \longmapsto \int_{\mathbb{R}} \omega$

is surjective, just take a bump fct $\lambda$ and compute $\int_{\mathbb{R}} \lambda \, dx$.

(c) $\underline{H^1_c(\mathbb{R}^2) = 0}$: $\alpha \in \Omega^1_c(\mathbb{R}^2)$ with $d\alpha = 0$. $H^1(\mathbb{R}^2) = 0 \Rightarrow \exists f \in C^\infty(\mathbb{R}^2)$: $df = \alpha$.
Take a curve $c$ from $P$ to $Q$ s.t. $c' \neq 0$ everywhere and $\mathrm{im}(c) \cap \mathrm{supp}(\alpha) = \emptyset$

Then $0 = \int_c \alpha|_c = \int_c df|_c \underset{\text{Stoke's}}{=} \int_{\partial c} f = f(Q) - f(P)$

$\Rightarrow \alpha \in \mathrm{im}(d_c^{(0)})$.

(d) $\underline{H^2_c(\mathbb{R}^2) \cong \mathbb{R}}$: We only have to show that $\omega \in \Omega^2_c(\mathbb{R})$ with $\int_{\mathbb{R}^2} \omega = 0$ is in the image of $d_c^{(1)}$.

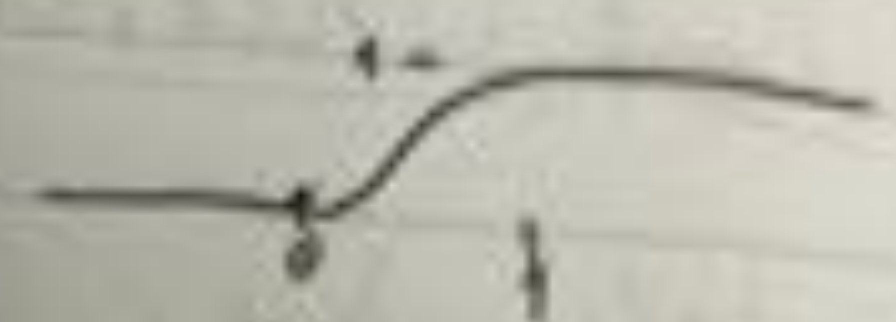
$\omega = a \, dx \wedge dy$. $\mathrm{supp}(a) \subseteq ]-R, R[^2$

Put: $A(x,y) := \int_{-R}^{x} a(t,y) \, dt$

Then $A = 0$ if $|y| \geq R$ or $x \leq -R$

- $\lambda$ a bump map supported in $]0,1[$ s.t. $\int_{\mathbb{R}} \lambda \, dt = 1$

$\Lambda(t) := \int_0^t \lambda(s)\, ds$



- $B(y) := \int_{-R}^{R} a(t,y)\, dt$

$B\, dy \in \ker(d^{(1)}_{c,\mathbb{R}})$

$(b) \Rightarrow \exists\, C \in \Omega^1_c(\mathbb{R}) : dC = B\,dy$

Then

$$\omega_0 := (A - \Lambda(x) B(y))\, dy - (\lambda(x) C(y))\, dx$$

$\in \Omega^1_c(\mathbb{R}^2)$, because

$\operatorname{supp}(\lambda(x) C(y)) \subseteq [-R,R]^2$

and $A(x,y) - \Lambda(x) B(y) = 0$ for $x \geq R$, so has supp $\subseteq [-R,R]^2$

$$
\begin{aligned}
d\omega_0 &= a\, dx \wedge dy - \lambda(x) B(x)\, dx \wedge dy \\
&\quad + \lambda(x)\, d(c\, dy) \\
&= a\, dx \wedge dy.
\end{aligned}
$$

<u>Proof (Theorem 5.2)</u>:

We show that for $n \geq 2$ and $k \geq 1$: $\quad H^k_c(\mathbb{R}^n) \simeq H^{k+1}_c(\mathbb{R}^{n+1})$

and $\quad H^1_c(\mathbb{R}^n) = 0$.

<u>$H^1_c(\mathbb{R}^n)$</u>: Take $\omega = a\, dx_1 \in \Omega^1_c(\mathbb{R}^n)$

such that $d\omega = 0$.
Now the argument is as in 5.3. (c).

<u>$H^k_c(\mathbb{R}^n) \simeq H^{k+1}_c(\mathbb{R}^{n+1})$</u>: $(k \geq 1,\ n \geq 2)$

We define maps

$$
\begin{aligned}
i_k &: \Omega^k_c(\mathbb{R}^n) \to \Omega^{k+1}_c(\mathbb{R}^{n+1}) \\
j_k &: \Omega^{k+1}_c(\mathbb{R}^{n+1}) \longrightarrow \Omega^k_c(\mathbb{R}^n).
\end{aligned}
$$

We consider a bump function $\lambda : \mathbb{R} \to [0,1]$ with integral 1.

- $\iota_k \left(\sum_I a_I \, d\underline{x}^I\right) := \sum_I a_I \, \lambda(x_{n+1}) \, dx^I \wedge dx_{n+1}$

Then
$$(d\iota_k)(a_I \, dx^I) = \lambda(x_{n+1}) \sum_{j \notin I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx^I \wedge dx_{n+1}$$

$$\iota_{k+1} d(a_I \, dx^I) = \iota_{k+1}\left(\sum_{j \notin I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx^I\right)$$

$$= \lambda(x_{n+1}) \sum_{j \notin I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx^I$$

$\Rightarrow \quad d\iota_k = \iota_{k+1} d \qquad$ (also for $k=0$)

- $j_{k+1}\left(\sum_I a_I(\underline{x}, x_{n+1}) \, dx^I + \sum_J b_J(\underline{x}, x_{n+1}) \, dx^J \wedge dx_{n+1}\right)$

$$:= \sum_J \left(\int_{-\infty}^{\infty} b_J(\underline{x}, s) \, ds\right) dx^J$$

Then $dj_{k+1} = j_{k+2} d$ (Exercise!) (also for $k=0$)

So we have
$$H^k_c(\mathbb{R}^n) \underset{j_{k+1}}{\overset{\iota_k}{\rightleftarrows}} H^{k+1}_c(\mathbb{R}^{n+1})$$

$$j_{k+1} \circ r_k = id_{\Omega_c^k(\mathbb{R}^n)}$$

We show that $r_k \circ j_{k+1}$ is homotopy equivalent to $id_{\Omega_c^{k+1}(\mathbb{R}^{n+1})}$

We need a homotopy operator $H$ s.t.

$$1 - r_k j_{k+1} = \pm d_c^{(k)} H_{k+1} \mp H_{k+2} d_c^{(k+1)}$$

$$H_{k+1} : \Omega_c^{k+1}(\mathbb{R}^{n+1}) \longrightarrow \Omega_c^k(\mathbb{R}^{n+1})$$

Take $\omega \in \Omega_c^{k+1}(\mathbb{R}^{n+1})$

$$\omega = \sum_I a_I(\underline{x}, x_{n+1})\, dx^I + \sum_J b_J(\underline{x}, x_{n+1})\, dx^J \wedge dx_{n+1}$$

$$H_{k+1}(\omega) := \sum_J \left( \int_{-\infty}^{x_{n+1}} b_J(\underline{x}, s)\, ds - \Lambda(x_{n+1}) \int_{-\infty}^{\infty} b_J(\underline{x}, s)\, ds \right) dx^J$$

Then

$$d_c^{(k)} H_{k+1}(a_I\, dx^I) - H_{k+2}\cancel{d_c^{(k+1)}}\Big(\sum_{j\notin I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx^I + \frac{\partial a_I}{\partial x_{n+1}} dx_{n+1}\wedge dx^I\Big)$$

$$= 0 - (-1)^{k+1} a_I\, dx^I$$

$$\iota_k\, j_{k+1}(a_I\, dx^I) = \iota_k(0) = 0.$$

End Lecture 25.04.2023

---

and

$$d_c^{(k)} H_{k+1}(b_J\, dx^J \wedge dx_{n+1}) - H_{k+2}\, d_c^{(k+1)}(b_J\, dx^J\wedge dx_{n+1})$$

$$= \sum_{j\notin J} \int_{-\infty}^{x_{n+1}} \frac{\partial b_J}{\partial x_j}(\underline{x}, s)\, ds\, dx_j \wedge dx^J$$

$$- A(x_{n+1}) \sum_{j\notin J} \int_{-\infty}^{\infty} \frac{\partial b_J}{\partial x_j}(\underline{x}, s)\, ds\, dx_j \wedge dx^J$$

$$+ b_J(\underline{x}, x_{n+1})\, dx_{n+1}\wedge dx^J - \underbrace{\lambda(x_{n+1})}_{\int_{-\infty}^{\infty} b_J\, ds} dx_{n+1}\wedge dx^J \longrightarrow$$

$$-\sum_{j\notin J}\int_{-\infty}^{x_{n+1}}\frac{\partial b_J}{\partial x_j}(x,s)\,ds\,dx_j\wedge dx^J\wedge dx_{n+1}$$

$$+\sum_{j\in J}\lambda(x_{n+1})\int_{-\infty}^{\infty}\frac{\partial b_J}{\partial x_j}(x,s)\,ds\,dx_j\wedge dx^J\wedge dx_{n+1}$$

$$=(-1)^k\left(b_J\,dx^J\wedge dx_{n+1}-\lambda(x_{n+1})\int_{-\infty}^{\infty}b_J\,ds\,dx^J\wedge dx_{n+1}\right)$$

$$=(-1)^k(1-i_k\,j_{k+1})(b_J\,dx^J\wedge dx_{n+1})$$

□

Theorem 5.4: Let $M$ be a mf. given by one global chart. Then

$$H_c^k(M)\simeq H_c^{k+1}(M\times\mathbb{R})\ \forall k\geq 0.$$

Proof: This is a corollary of the proof of Theorem 5.2. □

Corollary 5.5:
$$H_c^k(\mathbb{R}^{n-1}\times[0,\infty[)\cong\begin{cases}\mathbb{R}, & k=n\\ 0, & k\neq n\end{cases}$$

$k\geq 0,\ n\geq 1.$

Proof: By Theorem 5.4. we only need to compute the compactly supported cohomology for $n=1$.

$H_c^0([0,\infty[) = 0$, because $[0,\infty[$ is not compact, but connected.

$\underline{H_c^1([0,\infty[):}$ $\omega \in \Omega_c^1([0,\infty[)$

$$\omega = a\,dx \underset{\uparrow\, H^1([0,\infty[)=0}{=} db$$

$$\int_{[0,\infty[} \omega \underset{\substack{\uparrow \\ R>>1}}{=} \int_{[0,R]} \omega = b(R) - b(0).$$

~~So $\omega \in \operatorname{im}(d_c^{(0)}) \iff \int_{[0,\infty[} \omega = 0$.~~

~~Further $\exists\, \omega \in \Omega_c^1([0,\infty[) : \int_{[0,\infty[} \omega \neq 0$.~~

~~Thus $H_c^1([0,\infty[) \cong \mathbb{R}$~~
~~$[\omega] \longmapsto \int_{[0,\infty[} \omega$.~~

$\Rightarrow \omega \in \operatorname{im}(d_c^{(0)})$. $\square$

<u>Def 5.6:</u> Let $M$ be a mf. and $k \geq 0$ ($\partial M = \emptyset$)

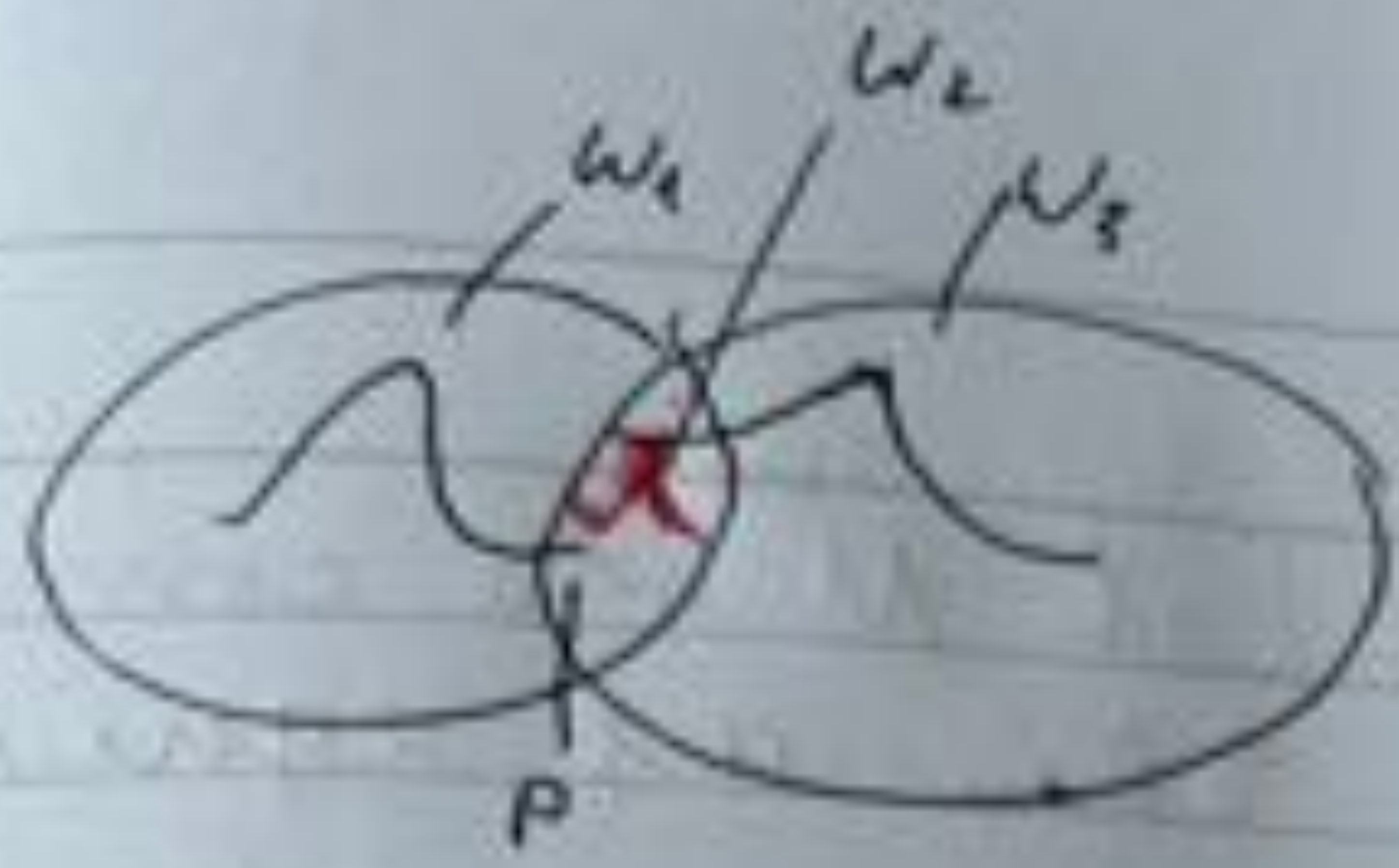
A differential $k$-form $\omega \in \Omega_c^k(M)$ is called a bump form if $\exists$ $(\varphi, U)$ a chart such that $\operatorname{supp}(\omega) \subseteq U$ and compact and $\omega = \lambda \, dx^I$ for some bump function $\lambda$.

<u>Lemma 5.7:</u> Let $M^m$, $m \geq 1$, be a connected mf ($\partial M = \emptyset$) and let $\omega_1, \omega_2$ be two non-zero bump $m$-forms. Then $\exists$ $d \in \mathbb{R}$: $[d\omega_1]_c = [\omega_2]_c$ in $H_c^m(M)$.

<u>Proof</u> (for orientable $M$): $BF_1(M) := \{ \omega \in \Omega_c^m(M) \mid \omega$ bump form $\int_M \omega = 1 \}$

$$BF_1(M)_p := \{ \omega \in BF_1(M) \mid p \in \operatorname{int}(\operatorname{supp}(\omega)) \}$$

For a chart $(\varphi, U)$ such that $\varphi(U)$ is an open ball all bump forms $\omega \in BF_1(M)_p$ are compactly cohomologous to each other because $H_c^m(U) \simeq \mathbb{R}$ via $\int_{\overline{U}}$.




$P \subseteq \mathrm{supp}(\omega_1)^\circ \cap \mathrm{supp}(\omega_3)^\circ \cap \mathrm{supp}(\omega_2)^\circ$

By transitivity all forms in $BF_1(M)_P$ are cohomologous to each other.
W.l.o.g. $\int \omega_1 = 1$. Take $P \in \mathrm{supp}(\omega_1)^\circ$.

$X = \{ Q \in M \mid \exists\, \omega \in BF_1(M)_Q :$

$\omega$ is compactly cohomologous to $\omega_1 \}$

$X$ is open and $M \setminus X$ is open and $X \neq \emptyset$.

$\Rightarrow M = X$ because $M$ is connected

$\square$

Exercise: Write the proof of 5.7. for a non-orientable $M$.

<u>Def 5.8</u>: Let $M^m$, $m \geq 1$, be a mf.
A <u>gallery</u> of <u>charts</u> is a sequence

$(\varphi_1, U_1), \ldots, (\varphi_\ell, U_\ell)$ of connected charts for $M$ such that

(i) $\forall i = 1, \ldots, \ell-1 \quad U_i \cap U_{i+1} \neq \emptyset$

(ii) $\forall i = 1, \ldots, \ell-1$: $D(\varphi_{i+1} \circ \varphi_i^{-1})$ has positive determinant at every point of $\varphi_i(U_i \cap U_{i+1})$.

<u>Lemma 5.9:</u> Let $M^m$ be a connected mf., $m \geq 1$. Then are equivalent

(1) $M$ is not orientable
(2) $\exists p \in M$: $\exists$ gallery $(\varphi_1, U_1), \dots, (\varphi_\ell, U_\ell)$ with $p \in U_1 = U_\ell$ s.t. $\det(D(\varphi_1 \circ \varphi_\ell^{-1}))$ is negative on $\varphi_\ell(U_\ell)$

(3) $\forall_{p \in M}$: $\exists$ gallery $(\varphi_1, U_1), \dots, (\varphi_\ell, U_\ell)$ with $p \in U_1 = U_\ell$ s.t. $\det(D(\varphi_1 \circ \varphi_\ell^{-1}))$ is negative.

<u>Proof:</u> Exercise. $\square$

<u>Theorem 5.10:</u> Let $M^m$, $m \geq 1$, be a connected mf. $\partial M = \emptyset$ Then

$$H_c^m(M) \cong \begin{cases} \mathbb{R}, & M \text{ orientable} \\ 0, & M \text{ not orientable} \end{cases}$$

Proof: 1) Take $p \in M$ and
~~Blah~~ $\omega_0 \in BF_1(M)_p$.

Take $[\omega]_c \in H^m_c(M)$ and a partition of unity $(\lambda_i)_{i\in I}$ such that $\varphi_i(U_i)$ is a ball in $\mathbb{R}^m$ or ~~in~~ half ball in $H_{\varphi_i}$.

- $\omega\lambda_i$ is cohomologous to $d_i\omega_0$ for some $d_i \in \mathbb{R}$ by Lemma 5.7.
- We have $\omega = \sum_{i\in I_0} \lambda_i \omega$ with $I_0$ finite, because supp$(\omega)$ is compact.

$$\Rightarrow \quad [\omega]_c = \sum_{i\in I_0} [\lambda_i \omega]_c$$
$$= \Big(\sum_{i\in I_0} d_i\Big)[\omega_0]_c$$

$$\Rightarrow \quad \dim_{\mathbb{R}} H^m_c(M) \leq 1.$$

2) Let $M$ be orientable.

$$\Phi : H^m_c(M) \longrightarrow \mathbb{R}$$
$$[\omega]_c \longmapsto \int_{(M,\sigma)} \omega \text{ is}$$

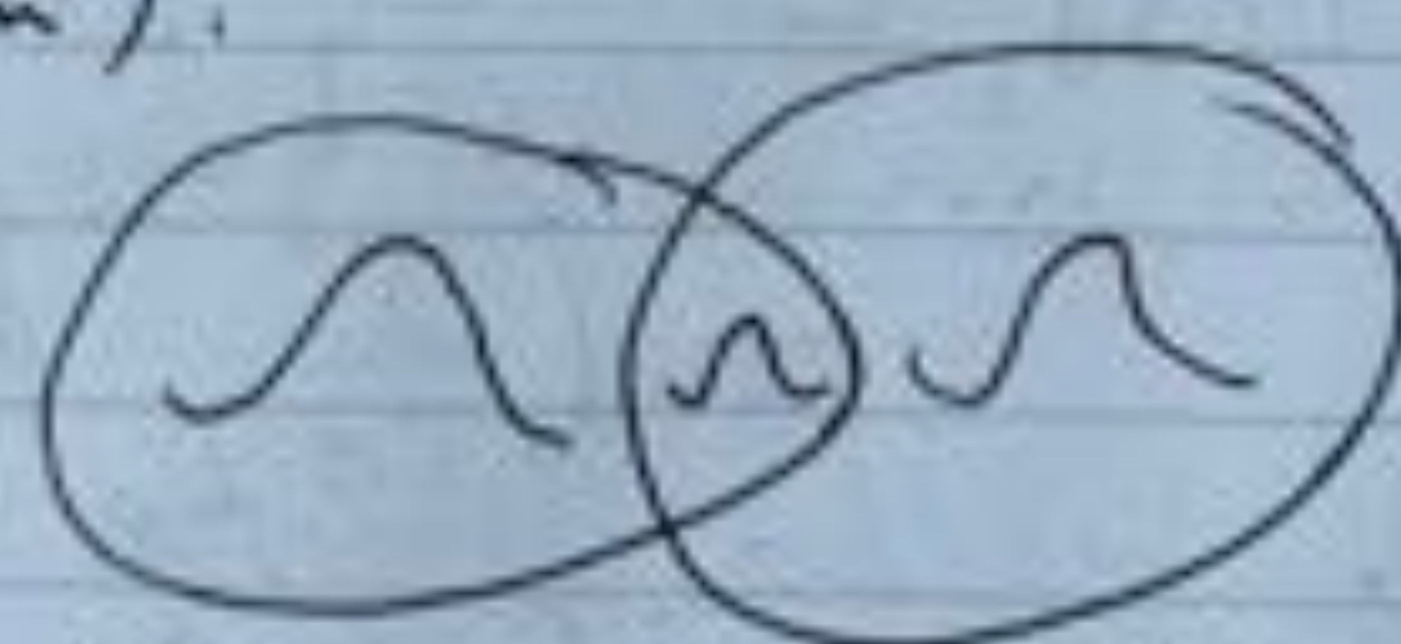
surjective, by dimension bijective.

3) Let $M$ ~~set~~ be not-orientable.

Take $p$ and $\omega_0$ as in 1).
Let $(\varphi_1, U_1), \dots, (\varphi_\ell, U_\ell)$ be a gallery with $p \in U_1 = U_\ell$ and $\det(D(\varphi_1 \circ \varphi_\ell^{-1})) < 0$, see Lemma 5.9.

<u>Claim:</u> $\exists \underset{\omega_0}{\overset{\shortparallel}{\omega_1}}, \omega_2, \omega_3, \dots, \omega_\ell$ : bump forms s.t. $\operatorname{supp}(\omega_i) \subseteq U_i$ such that $\omega_i$ is compactly cohomologous to $\omega_0$ for all $i = 1, \dots, \ell$ and $\omega_i\left(\frac{\partial}{\partial x_j^{(i)}}\right) \geq 0$

<u>Proof (Claim):</u>



$U_i$ $\qquad$ $U_{i+1}$

Exercise $\square$

Then $[\omega_0]_c = [\omega_\ell]_c$

and $I_{(\varphi_1, U_1)}(\omega_\ell) = -I_{(\varphi_\ell, U_1)}(\omega_1)$

$\leq 0$, because $\omega_e\left(\frac{\partial}{\partial x_1^{(0)}}, \dots, \frac{\partial}{\partial x_e^{(0)}}\right) \geq 0$

$\Rightarrow [\omega_0]_c = [\omega_e]_c = I_{(q_1, U_1)}(\omega_e)\,[\omega_0]_c$

$\Rightarrow [\omega_0]_c = 0$ $\square$

End Lecture 28.04.2023

Corollary 5.11: Let $M^m$ be a connected compact mf. ($\partial M = \emptyset$) Then

$$H^m(M) \cong \begin{cases} \mathbb{R}, & \text{if } M \text{ is orientable} \\ 0, & \text{else} \end{cases}$$

We want to compare cohomology and compactly supported cohomology.

Def 5.12: Let $M^m$ be a mf without boundary. An open cover $(U_i)_{i\in I}$ of is called good if $\forall_{\emptyset \neq J} \subseteq I$ finite we have $\bigcap_{i\in J} U_i$ is diffeomorphic to $\mathbb{R}^m$.

A manifold $M$ is called of finite type if it has a finite good cover.