Prop. 5.13: (Bott-Tu, Diff. forms in algebraic topology. Thm. 5.1)

Let M be a mf with $\partial M = \emptyset$. Then M has a good cover. Further M is of finite type if M is compact.

It uses "Riemannian Geometry" by Manfredo Do Carmo. (See Prop. 4.2. page 76 there)

This statement is part of Riemannian geometry. We do not give a proof here.

Idea. (M, g) Riemannian mf.
$P, Q \in M$.

$$d(P,Q) := \inf \Big\{ \underbrace{\int_a^b |c'(t)|_g \, dt}_{\ell(c)} \;\Big|\; c \text{ a smooth curve from } P \text{ to } Q \text{ with } c'(t) \neq 0 \ \forall t \Big\}$$

A curve c is called minimizing if $d(P,Q) = \ell(c)$.
$X \subseteq M$ is called convex if $\forall P, Q \in X$ $\exists!$ minimizing curve c from P and Q parametrized by length

and the image of this curve lies in X.
Now ~~the~~ take a cover consisting of convex neighborhoods of points.

□

Prop. 5.14: (Mayer-Vietoris sequence for cohomology with compact support)

Let $M = U \cup V$ be a mf. with U, V open subsets.
Then

1) The inclusion $\iota_U^M : U \to M$ defines a map

$$(\iota_U^M)_* : \Omega_c^*(U) \to \Omega_c^*(M)$$

just via extending by 0.

2) The sequence

$$0 \to \Omega_c^*(U \cap V) \xrightarrow{\alpha} \Omega_c^*(U) \oplus \Omega_c^*(V) \xrightarrow{\beta} \Omega_c^*(U \cup V) \to 0$$

with $\alpha := (\iota^{U}_{U\cap V})_* \oplus (\iota^{V}_{U\cap V})_*$

and $\beta := (\iota^{M}_{U})_* - (\iota^{M}_{V})_*$.

is exact, and α and β are chain maps.

3) $\exists$ long exact sequence (Mayer - Vietoris sequence) for cohomology groups with compact support

$$\begin{tikzcd} 0 \arrow[r] & H^0_c(U\cap V) \arrow[r, "\alpha^{(0)}"] & H^0_c(U)\oplus H^0_c(V) \arrow[r, "\beta^{(0)}"] & H^0_c(U\cup V) \arrow[dll, "\delta^{(0)}"'] \\ & H^1_c(U\cap V) \arrow[r, "\alpha^{(1)}"] & H^1_c(U)\oplus H^1_c(V) \arrow[r, "\beta^{(1)}"] & H^1_c(U\cup V) \arrow[dll, "\delta^{(1)}"'] \\ & \cdots \end{tikzcd}$$

__Proof:__ Exercise $\square$

Lemma 5.15: (Five lemma)

Consider the following commutative diagram of group homomorphisms

$$\begin{array}{ccccccccc} G_1 & \to & G_2 & \to & G_3 & \to & G_4 & \to & G_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ H_1 & \to & H_2 & \to & H_3 & \to & H_4 & \to & H_5 \end{array}$$

with exact rows.

(i) If $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ are isomorphisms then α_3 is an isomorphism.

(ii) In (i) it is enough for α_1 to be surjective and α_5 to be injective.

Proof: diagram chase $\square$

Example 5.16: (a) $\mathbb{R}^n$, $n \geq 1$.

We have $H^k_c(\mathbb{R}^n) \simeq \begin{cases} \mathbb{R}, & k = n \\ 0, & k \neq n \end{cases}$

$\simeq H^{n-k}(\mathbb{R}^n)$, $0 \leq k \leq n$.

(b) $n \geq 1$, $H^k_c(S^n) = H^k_{dR}(S^n) = \begin{cases} \mathbb{R}, & k = n, 0 \\ 0, & \text{else} \end{cases}$

$$\simeq H^{n-k}_{dR}(S^n), \quad 0 \leq k \leq n$$

This is not a coincidense.

We have the following form.
Let M be a mf. oriented $\partial M = \emptyset$.

Consider $(*) \int_M : H^k_c(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$

$$([\omega]_c, [\xi]) \mapsto \int_M \omega \wedge \xi.$$

Theorem 5.17: (Poincaré duality)

Let M^m be an orientable mf without boundary ~~...~~ s.t. M admits a finite good ~~...~~ cover. Then the pairing $(*)$ is non-degenerate, i.e. $(*)$ induces

$$(H^k_c(M))^* \simeq H^{m-k}_{dR}(M)$$

for all $0 \leq k \leq m$.

Proof: Part 1: The pairing is well-defined: $[\omega_1]_c = [\omega_2]_c \in H^k_c(M)$
$\Rightarrow \quad \omega_1 - \omega_2 = d\alpha, \ \alpha \in \Omega^{k-1}_c(M)$

For $\varrho \in \Omega^{m-k}(M)$ with $d\varrho = 0$ we have

$$d\alpha \wedge \varrho = d(\alpha \wedge \varrho) - (-1)^{\deg(\alpha)} \alpha \wedge \underbrace{d\varrho}_{0}$$

$$\Rightarrow \quad \int_M d\alpha \wedge \varrho = 0 \text{ by Stoke's Thm,}$$

because $\partial M = \emptyset$.
Similarly the case $[\varrho_1] = [\varrho_2] \in H^{m-k}(M)$

Part 2: Let M have a finite good cover $M = U_1 \cup \dots \cup U_\ell \quad \ell \in \mathbb{N}$.
We prove the theorem by induction on ℓ.

$\ell = 1$: $\quad M \simeq \mathbb{R}^m$, by definition of a good cover. (Exercise.)

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$\ell > 1$: $\quad N := U_1 \cup \dots \cup U_{\ell-1}, \ V := U_\ell$.